ON THE REPRESENTATION THEORY OF THE CHEVALLEY GROUPS

BY

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Abstract

In this thesis we investigate the representation theory of the following classes of groups:- $GL(m,q)$, $SL(m,q)$, $Sp(2m,q)$, $\Omega(m,q)$ and their quotients by their centres.

In the first part we prove the following theorem:-

<u>Theorem</u> If G is a group of one of the above types, the only possible defect groups of G are the Sylow p-subgroup and the trivial subgroup.

We then use this to obtain a complete determination of the p-block structure in each case.

In the second part we construct and investigate the properties of a family of irreducible characters for the above groups.

Preface This thesis presents the results of two investigations into the character theory of the non-exceptional Chevalley groups. In both cases the methods used enable us at the same time to consider the groups $GL(\ell, q)$, $PGL(\ell, q)$ and the covering groups of the afore-mentioned Chevalley groups. I do not doubt but that the methods employed are also applicable to the exceptional Chevalley groups and to the Steinberg groups, but since in both cases we are reduced, after a certain amount of general theory, to a consideration of particular cases, I felt that enough was enough.

Each of the two investigations is self-contained and is here presented as such. This means in particular that the references for chapter one are to be found at the end of that chapter rather than at the end of the thesis, and that each of the two chapters has its own introduction.

Except where explicitly stated the results presented are my own. Having said which, I should add that this research was carried out under the supervision of Professor J. A. Green, and that I am considerably indebted to him for much help and encouragement. Without it it is doubtful whether either of these investigations would have been completed. I should also like to thank my fellow students, R. J. Clarke and D. L. Johnson for many helpful

conversations on these and other problems.

Finally, I should like to express my gratitude to the Science Research Council for the grant which enabled this work to be carried out.

The Block Structure of Certain Classical Groups

Introduction Let G be a finite group and k a field of characteristic p.* Then a p-block - hereafter referred to simply as a block - is an indecomposable two-sided ideal of the group algebra kG. These ideals play a large role in the modular representation theory of finite groups. To each block is assigned a p-subgroup of G (10), which is known as the defect group of the block. There are several ways of defining defect groups; the original one, due to Brauer, may be found in (6), but we shall be using the definition of Rosenberg (12) and Green(9). The defect of a block is simply the exponent of the order of its defect group. Thus a block has maximal defect if its defect group is the Sylow p-subgroup of G, and defect zero if its defect group is trivial.

In (14) Steinberg proved that the Chevalley groups $L(p^a)$ have exactly one block of defect zero. In (4) and (5) Curtis proved a similar theorem for a larger class of groups. We shall be concerned with a complete determination of the block structure of the non-exceptional Chevalley groups defined over finite fields of characteristic p. Ree (11) showed that these are quotients of certain classical groups, and, as a result

* In what follows we shall assume that k is a splitting field for 3. G.

of the methods used, we shall also obtain structure theorems for the blocks of these latter groups.

The idea of the proof is to show that for such groups the only possible defect groups are the Sylow p-subgroup and the trivial subgroup. Once this has been done the results follow in a fairly straightforward manner. At this point I should like to express my gratitude to Professor Green for showing me an early version of (10), which enabled me to realise that this method of attack - i.e. via the defect groups - was feasible. His theorem does not yield the final result, but it reduces the possibilities to such a degree that the proof may then be completed by means of the earlier characterisations of the defect group which are due to Brauer (1).

For the proofs of subsequently quoted facts about Lie algebras and Chevalley groups we refer the reader to (2), (3), or (11).

<u>Notation and Preliminaries</u> Let $q = p^a$, and let $GL(m, q)$ be the group of non-singular $m \times m$ matrices having entries in the field $GF(q)$. Let $SL(m, q)$ be the subgroup of $GL(m, q)$ which consists of those matrices of determinant one. Let $Sp(2m, q)$ be the subgroup of $GL(2m, q)$ which consists of those matrices T for which $T'AT = A$, where

$A_1 = \begin{bmatrix} & I_m \\ -I_m & \end{bmatrix}$.

Let $\Omega(2m,q)$ be the commutator subgroup of the subgroup of matrices of $GL(2m,q)$ which consists of those matrices T for which $T'A_2T = A_2$ where $A_2 = \begin{bmatrix} & I_m \\ I_m & \end{bmatrix}$.

Let $\Omega(2m+1,q)$ be the commutator subgroup of the group of matrices of $GL(2m+1,q)$ which consists of those matrices T for which $T'A_3T = A_3$ where $A_3 = \begin{bmatrix} 2 & & \\ & & I_m \\ & I_m & \end{bmatrix}$.

Ree's results in (11) were that $A_m(q) \cong PSL(m+1,q)$; $B_m(q) \cong P\Omega(2m+1,q)$; $C_m(q) \cong PSp(2m,q)$; and $D_m(q) \cong P\Omega(2m,q)$.

Let G be a group of one the following types;- $GL(m,q)$, $SL(m,q)$, $Sp(2m,q)$, $\Omega(m,q)$. Let P be a Sylow p-subgroup of G, and let B be the normaliser in G of P. Let H be the subgroup of G which consists of the diagonal matrices.

We now define a subgroup N and shall give separate definitions for each group G. Basically N is the subgroup of monomial matrices, but this fact requires proof and is not material to our purposes.

<u>(i) $GL(m,q)$ and $SL(m,q)$</u>

Let $n_{i,i+1}$ be the matrix

$$
\begin{array}{cc}
\left[\begin{array}{ccc|cc|ccc}
1 & & & & & & & \\
 & 1 & 0 & & & 0 & & 0 \\
 & 0 & \ddots & & & & & \\
 & & 1 & & & & & \\
\hline
 & & & 0 & 1 & & & \\
 & & & -1 & 0 & & & \\
\hline
 & & & & & 1 & & 0 \\
 & & & & & & \ddots & \\
 & 0 & & & 0 & & & \\
 & & & & & 0 & & 1
\end{array}\right]
&
\begin{array}{c}
1 \\ \cdot \\ \\ i-1 \\ i \\ i+1 \\ i+2 \\ \cdot \\ \\ m
\end{array}
\end{array}
$$

Then $N = \{Gp.\ H,\ n_{i,i+1};\ i = 1,2,\ldots\ldots m-1\}$

<u>(ii) Sp(2m,q)</u>

Let $n_{i,i+1}$ be the matrix

$$
\begin{array}{cc}
\left[\begin{array}{c|c|c|c|c|c}
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \begin{matrix}0 & 1 \\ 1 & 0\end{matrix} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \begin{matrix}0 & 1 \\ 1 & 0\end{matrix} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}
\end{array}\right]
&
\begin{array}{c}
1 \\ \cdot \\ \\ i \\ i+1 \\ \\ m \\ -1 \\ \\ -i \\ -i-1 \\ \\ -m
\end{array}
\end{array}
$$

Then $N = Gp.\{H,\ n_{m,-m},\ n_{i,i+1};\ i = 1,2,\ldots\ldots m-1\}$

where $n_{m,-m}$ is the matrix

$$\left[\begin{array}{c|c|c|c}
\begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & & 0 & \\ \hline
 & 0 & & 1 \\ \hline
0 & & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & \\ \hline
 & -1 & & 0
\end{array}\right]$$

(iii) $\Omega(2m, q)$

Let $n_{i,i+1}$ be the matrix defined above in case (ii). Let $n_{m,-m}$ be the matrix

$$\left[\begin{array}{c|c|c|c}
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & & 0 & \\ \hline
 & 0 & & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \\ \hline
0 & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \\ \hline
 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & & 0
\end{array}\right]$$

Then $N = \mathrm{Gp.}\left\{H\ ,\ n_{m,-m},\ n_{i,i+1}\ ;\ i = 1,2,\ldots.m-1\right\}$

(iv) $\Omega(2m+1, q)$

Let $n_{i,i+1}$ and $n_{m,-m}$ be the $2m+1 \times 2m+1$ matrices whose first row and column are zero except for a 1 in the top left-hand corner, and whose remaining minors are the matrices of case (ii), save that in $n_{m,-m}$ the -1 of case (ii) is replaced by a 1.

Then $N = \mathrm{Gp.}\left\{H\ ,\ n_{m,-m}\ ,\ n_{i,i+1}\ ;\ i = 1,2,\ldots,m-1\right\}$

The subgroups B and N form a (B?N) pair in G. For a definition and an account of the properties of a (B,N) pair we refer the reader to (15). The properties that we shall need are as follows:-

(a) $G = BNB$

(b) $B = HP$

(c) $B \cap N = H$

(d) $H \lhd N$. $\quad N/_H \cong W$, the Weyl group of G.

For each element w_i of W we choose an element n_i of N such that $n_i \mapsto w_i$ under the above isomorphism. Thus the n_i are a set of coset representatives for H in N, and hence it follows from (a) that $G = \bigcup_i Bn_iB$. This brings us to our final property which is

(e) $Bn_iB \cap Bn_jB = \emptyset$ if $i \neq j$.

Let Γ be the GxG-module whose k-basis is the set of elements of G, and whose action is given by

$$\gamma\,(g_1, g_2) = g_1^{-1}\gamma g_2.$$

The indecomposable GxG-components, Γ_i, of Γ correspond to the blocks, $\mathcal{B}_i$, of G; and if the corresponding defect group is D_i, the module Γ_i has vertex $\Delta(D_i)$. (See (9))

§1

In this section we show how the (B,N) pair structure of G may be combined with the work of Green to produce a very powerful restriction on the possible defect groups of G.

<u>Lemma 1</u> $\Gamma_{P\times P} = \bigoplus_{\substack{n_i \\ h_j \in H}} kPn_i h_j P$

<u>Proof</u> Since $B = HP$, $P \lhd B$, $H \lhd N$, and $G = \bigcup_i Bn_i B$ ~~Pn_ih_jP~~,
it follows that $G = \bigcup_{n_i, h_j \in H} Pn_i h_j P$.
The result now follows from lemma 2.

<u>Lemma 2</u> $Pn_i h_j P \cap Pn_k h_m P = \emptyset$ unless $i = k$ and $j = m$.

<u>Proof</u> Suppose $i \neq k$.
Then since $Pn_i h_j P \leq Bn_i B$ and $Pn_k h_m P \leq Bn_k B$ the result follows from property (e) above.
Now suppose that $i = k$, but $j \neq m$.

$$|Pn_i B| = \frac{|P|\,|B|}{|B \cap P^{n_i}|} \qquad (P^{n_i} = n_i^{-1} P n_i)$$

$$|Pn_i h_j P| = \frac{|P|^2}{|P \cap P^{n_i h_j}|} = \frac{|P|^2}{|P \cap P^{n_i}|}$$

But since $B = HP$ and H is a p'-group, it follows that $B \cap P^{n_i} = P \cap P^{n_i}$. Therefore the number of double cosets $Pn_i h_j P$ in the set $Pn_i B$ is equal to $\frac{|B|}{|P|} = |H|$.

<u>Corollary</u> $(\Gamma_i)_{P\times P} \cong \bigoplus kPy_m P$ where the y_m are a subset of the $n_i h_j$.

<u>Lemma 3</u> The vertex of $(PyP)_{P\times P}$ is $\Delta\,(P \cap P^{y})^{(1,y)}$

<u>Proof</u> $y(p_1, p_2) = y$ iff $p_1^{-1} y p_2 = y$

iff $p_2 = y^{-1} p_1 y$

iff $(p_1, p_2) \in \{(p, p^y);\ p \in P \cap P^{y^{-1}}\}$

Therefore $(PyP)_{P\times P} \cong \left(k_{\Delta(P\cap P^{y^{-1}})^{(1,y)}}\right)^{P\times P}$

The result now follows from (9) lemma 2.3a.

Hence by a theorem of J. A. Green ((8), corollary to theorem 6) the vertex of Γ is $\Delta(P \cap P^{n})$ for some $n \in N$.

We can, however, say more because of the following theorem. This is theorem 3 of (10).

<u>Theorem 1</u> (J. A. Green) Let G be any finite group and let P be a Sylow p-subgroup of G. Then If $D \leq P$ is a defect group of G, there exists $y \in C_G(D)$ such that $D = P \cap P^y$.

We now show how, for the particular groups under consideration, this theorem may be strengthened still further. We have already shown that the only possible defect groups are of the form $P \cap P^{n_i}$. Since the centre of G is a p'-group, P has the same structure as the Sylow p-subgroup of the corresponding Chevalley group. That is to say by utilising, in the obvious way, the homomorphisms defined by Ree (11) we can define root subgroups of P such that the relations which hold in the Sylow p-subgroup of the Chevalley group also hold in P. Thus for each positive root r of L and for each $t \in GF(q)$ we have a matrix $x_r(t)$ such that

$$(1.1) \quad P \cap P^{n_i} = \text{Gp.}\left\{ x_r(t) \; ; \; t \in GF(q), \; r > 0 , \; w_i(r) > 0 \right\}$$

(1.2) $x_s(u)^{-1} x_r(t)\, x_s(u)\, x_r(t)^{-1} = \prod_{i,j} x_{ir+js}(C_{ij,rs}\, t^i u^j)$

where i, j run over all pairs of positive integers for which $ir+js$ is a root and the numbers $C_{ij,rs}$ are certain integers.

We shall define the $x_r(t)$ more explicitly in § 3. For the present it is sufficient that they exist and have the properties stated.

Theorem 1 shows us that if $D = P \cap P^n$ is a defect group there exist $p_1, p_2 \in P$ such that $p_1 n p_2 \in C_G(D)$ and $D = P \cap P^{p_1 n p_2}$. It is the point of the next two lemmas to show that this implies that n normalises each root subgroup of D.

<u>Lemma 4</u> $p_1 n p_2 \in C_G(P \cap P^{p_1 n p_2})$ implies that $p_2 p_1 n \in C_G(P \cap P^n)$.

<u>Proof</u> $P \cap P^{p_1 n p_2} = P \cap P^{n p_2} = (P \cap P^n)^{p_2}$

$$(P \cap P^n)^{p_2} = \{ p_2^{-1} \pi p_2 \;;\; \pi \in P \cap P^n \}$$

Let π be any element of $P \cap P^n$.

Then $p_1 n p_2 \cdot p_2^{-1} \pi p_2 \cdot p_2^{-1} n^{-1} p_1^{-1} = p_2^{-1} \pi p_2$

Therefore $p_2 p_1 n . \pi . n^{-1} p_1^{-1} p_2^{-1} = \pi$

Therefore $p_2 p_1 n \in C_G(P \cap P^n)$ as required.

<u>Lemma 5</u> Let n be any element of N, and let w be the corresponding element of W.

Let $\Pi_w = \{ r \;;\; r > 0 \,,\; w(r) > 0 \}$

Then $pn \in C_G(P \cap P^n)$ implies that $w(r) = r$ for every $r \in \Pi_w$.

<u>Proof</u> $P_n P^n = \mathrm{Gp.}\left\{ x_r(t) \; ; \; r \in \Pi_w , \; t \in GF(q) \right\}$

Thus our centraliser condition is

$$n^{-1} p^{-1} x_r(t)\, pn = x_r(t) \quad \text{for every } r \in \Pi_w.$$

Hence $\quad p^{-1} x_r(t)\, p = n\, x_r(t)\, n^{-1}$ for every $r \in \Pi_w$ (1.3)

It follows from (1.2) that we can write $p^{-1} x_r(t)\, p$ in the form $x_r(t)\, x_{s_1}(t_1) \, \ldots\ldots\ldots \, x_{s_\ell}(t_\ell)$ where for each i, $s_i = r +$ some positive root, and $r < s_1 < s_2 \ldots\ldots < s_\ell$

It is further true that this expression is the only one in which the ordering of the roots is respected.

However $\quad n\, x_r(t)\, n^{-1} = x_{w(r)}(at)$ for some a.

Hence by the afore-mentioned uniqueness $r = w(r)$.

~~Corollary If $D = P_n P^n$ is a defect group of G, n centralises~~ D.

§2

In this section we look at the root systems of the Lie algebras of type A_m, B_m, C_m, D_m, and find those w for which

(2.1) $\quad w(r) = r \quad$ for every $r \in \Pi_w$.

A_m: In Euclidean space take orthogonal vectors $a_0, a_1, \ldots\ldots a_m$. Then the fundamental roots of A_m are given by $\quad q_i = a_{i-1} - a_i$ for $i = 1,2,\ldots\ldots m$.

Thus the set of positive roots is $\left\{ a_i - a_j \; ; \; i < j \right\}$

Let u_i be the linear map which interchanges a_{i-1} and a_i, but which leaves the remaining a_j fixed.

Then $W = Gp. \{u_1, u_2, \ldots\ldots\ u_m\}$

Thus W is the full permutation group on the symbols $\{0, 1, 2, \ldots\ldots\ m\}$.

Suppose that w satisfies (2.1). Then there exists an a_j such that $w(a_j) = a_0$.

Case 1 $j = 0$

Consideration of the positive roots $a_0 - a_i$ for $i = 1, 2, \ldots\ldots\ m$ shows that $w(a_i) = a_i$ for every i.

Case 2 $j \neq m$, $j \neq 0$

Then $w(a_j - a_m) = a_0 - a_{w(m)}$ which contradicts (2.1).

Case 3 $j = m$

Consider $w(a_0)$. If $w(a_0) \neq a_m$ there exists $n \geq 1$ such that $w(a_n) = a_m$. Now consideration of $w(a_0 - a_n)$ gives a contradiction.

Hence either w is the identity permutation, or $w(a_0) = a_m$, $w(a_m) = a_0$.

Suppose the latter is the case. Then since $a_1, \ldots\ldots\ a_{m-1}$ form a root system of type A_{m-2}, we may repeat the above argument. Thus we have

Lemma 6 If L is a Lie algebra of type A_m, the w which satisfy (2.1) are precisely those which, when written as permutations, have the form

$$\begin{pmatrix} 0 & 1 & \ldots\ldots & j & \big| & j+1 & \ldots\ldots & m-j-1 & \big| & m-j & \ldots\ldots & m \\ m & m-1 & \ldots & m-j & \big| & j+1 & \ldots\ldots & m-j-1 & \big| & j & \ldots\ldots & 0 \end{pmatrix}$$

where j is any integer in the range 0 to m.

B_m: In Euclidean space take orthogonal vectors $a_1, a_2, \ldots\ldots a_m$. Then the fundamental roots of B_m are given by

$$q = a_i - a_{i+1} \text{ for } i = 1,2,\ldots m-1$$

$$q = a_m$$

The positive roots are

$$a_i, \text{ for } i = 1,2,\ldots\ldots m$$

$$a_i + a_j, \text{ for } i = j$$

$$a_i - a_j, \text{ for } i < j$$

For $i = 1,2,\ldots\ldots m-1$ let u_i be the linear map which interchanges a_i and a_{i+1}, but which leaves the remaining a_j fixed.

Let u_m be the linear map defined by

$$a_m \to -a_m$$

$$a_i \mapsto a_i \quad i \neq m$$

Then $W = \text{Gp.}\{u_1, u_2, \ldots\ldots u_m\}$.

Thus W permutes the a_i and changes the sign of any number of them.

C_m: Define $q_1, q_2, \ldots\ldots q_{m-1}$ as for B_m; but let $q_m = 2a_m$.

The positive roots are

$$a_i + a_j$$

$$a_i - a_j, \text{ for } i < j$$

Define $u_1, \ldots\ldots u_m$ as for B_m. Thus W is the same in both cases.

Suppose that w satisfies (2.1). Since a_i $(2a_i)$ is a positive root of B_m (C_m), $w(a_i) = a_j$ implies that $i = j$. Thus there are just two cases to be considered.

<u>Case 1</u> There exists an a_i such that $w(a_i) = a_i$

Let s be the least such i, and suppose $t > s$ is such that $w(a_t) \neq a_t$.

Consider the roots $a_s - a_t$ and $a_s + a_t$. (2.1) implies that the image under w of each of these roots must be negative. However $w(a_s - a_t) < 0$ implies that $w(a_t) = a_j$ for some $j < s$; while $w(a_s + a_t) < 0$ implies that $w(a_t) = -a_j$ for some $j < s$.

Therefore $t > s$ implies that $w(a_t) = a_t$.

Thus if $s = 1$ all the roots are fixed. Hence we may suppose that $s > 1$ and let $u < s$. Since $u < s$, $w(a_u + a_s)$ is not equal to $a_u + a_s$. Therefore $w(a_u + a_s) < 0$, and so $w(a_u) = -a_{w(u)}$.

Now consider the roots $a_u - a_v$ for $u < v < s$.

$$w(a_u - a_v) = -\left(a_{w(u)} - a_{w(v)}\right)$$

And so to comply with (2.1) we must have either $w(u) < w(v)$, or $w(u) = v$, $w(v) = u$.

For any given u, the latter condition can only occur for one v. ~~Thus if $u > 2$, the former condition must hold. Hence from this case we obtain two possibilities~~

Thus we may partition the indices less than s into three sets

$$\mathcal{X}_w = \{u\ ;\ w(a_u) = -a_{u+1}\}$$
$$\mathcal{Y}_w = \{u\ ;\ w(a_u) = -a_{u-1}\}$$
$$\mathcal{Z}_w = \{u\ ;\ w(a_u) = -a_u\}$$

and u belongs to $\mathcal{X}_w$ if and only if u+1 belongs to $\mathcal{Y}_w$
Note in particular that no two consecutive indices can belong to $\mathcal{X}_w$.

Case 2 $w(a_i) = -a_i$ for every i in the range 1 to m
The only positive roots which need any consideration are those of the form $a_u - a_v$ for $u<v$. The argument is now that of the second half of case one. In fact this is simply case one with $s = m+1$.

Hence we have proved

Lemma 7 If $\mathbb{L}$ is a Lie algebra of type B_m or C_m, the w which satisfy (2.1) fall into classes viz .

(i) there exists an s in the range 1 to m+1 such that

$$w(a_t) = a_t \text{ for } t \geqslant s$$
$$w(a_u) = -a_u \text{ for } u < s$$

(ii) there exists an s in the range 3 to m+1 such that

$$w(a_t) = a_t \text{ for } t \geqslant s$$
$$w(a_u) = -a_{u+1}\ ,\ u \in \mathcal{X}_w$$

$w(a_u) = -a_{u-1}$, $u \in Y_w$

$w(a_u) = -a_u$ for $u \in Z_w$

D_m: In Euclidean space take orthogonal vectors $a_1, a_2, \ldots\ldots a_m$. Then the fundamental roots of D_m are given by

$$q_i = a_i - a_{i+1} \text{ for } i = 1,2,\ldots\ldots m-1$$
$$q_m = a_{m-1} + a_m$$

The positive roots are

$$a_i + a_j \quad i \neq j$$
$$a_i - a_j \quad i < j$$

For $i = 1,2,\ldots\ldots m-1$ let u_i be the linear map which interchanges a_i and a_{i+1}, but which leaves the remaining a_j fixed.

Let u_m be the linear map defined by

$$a_m \to -a_{m-1}$$
$$a_{m-1} \to -a_m$$
$$a_i \to a_i, \quad i < m-1$$

Then $W = \mathrm{Gp.}\{u_1, u_2, \ldots\ldots u_m\}$

Thus W permutes the a_i and changes the sign of an even number of them.

Suppose that w satisfies (2.1). We can no longer rule out the possibility that there exists an i such that $w(a_i) = a_j$ for some $j \neq i$.

Suppose there are at least three such a_i.

Denote them a_d, a_e, a_f.

Then $w(a_d + a_e) \gtrless 0$, and so we must have $w(a_d) = a_e$

$w(a_e) = a_d$. But consideration of $w(a_e + a_f)$ now yields a contradiction.

Therefore there can be at most two such a_i.

Equally clearly if such an a_i exists we can not have an a_j such that $w(a_j) = a_j$.

Thus there are four cases to be considered.

<u>Case 1</u> There exists an a_i such that $w(a_i) = a_i$.

<u>Case 2</u> $w(a_i) = -a_{w(i)}$ for every i in the range 1 to m.

These cases are the same as the corresponding cases for B_m.

<u>Case 3</u> $w(a_s) = a_{w(s)}$, $s \neq w(s)$; $w(a_i) = -a_{w(i)}$, if $i \neq s$

If $s \neq m$ there exists $j > s$. Then $w(a_s - a_j)$ is a positive root other then $a_s - a_j$ which is a contradiction.

Therefore $s = m$.

Since $w(m) \neq m$, there exists an r such that $w(a_r) = -a_m$.

But now $w(a_m + a_r) = a_{w(m)} - a_m$ which is a positive root.

Therefore this case can not occur.

<u>Case 4</u> There exists r, s such that $w(a_r) = a_s$, $w(a_s) = a_r$ while $w(a_i) = -a_{w(i)}$ for $i \neq r, s$.

Let $s > r$.

As in case 3 we must have $s = m$ and similarly that $r = m-1$.

We now apply the argument of case 1 to the set of roots $a_i - a_j$ for $i < j < m-1$. Thus for D_m we get two additional possibilities. Hence we have proved,

Lemma 8 If L is a Lie algebra of type D_m, the w which satisfy (2.1) fall into four classes , viz

(i) there exists an s in the range 1 to m+1 such that $w(a_t) = a_t$ for $t \geq s$, $w(a_u) = -a_u$ for $u < s$

(ii) there exists an s in the range 3 to m+1 such that $w(a_t) = a_t$ for $t \geq s$; the indices less than s are partitioned into three sets $\mathcal{X}_w, \mathcal{Y}_w, \mathcal{Z}_w$ such that $w(a_u) = -a_{u+1}$ for $u \in \mathcal{X}_w$, $w(a_u) = -a_{u-1}$ for $u \in \mathcal{Y}_w$ and $w(a_u) = -a_u$ for $u \in \mathcal{Z}_w$

(iii) $w(a_m) = a_{m-1}$, $w(a_{m-1}) = a_m$

$w(a_i) = -a_i$ for $i \leq m-2$

(iv) $w(a_m) = a_{m-1}$, $w(a_{m-1}) = a_m$, the remaining indices being partitioned into three sets such that $w(a_u) = -a_{u+1}$ for $u \in \mathcal{X}_w$, $w(a_u) = -a_{u-1}$ for $u \in \mathcal{Y}_w$, and $w(a_u) = -a_u$ for $u \in \mathcal{Z}_w$

§ 3

In this section we complete the proof of the following theorem

Theorem 2 If G is a group of the type under discussion the only possible defect groups of G are the Sylow p-subgroup, P, and the trivial subgroup.

In the previous two sections we have shown that a defect group must have the form $P_\cap P^n$, and that n must have the property that the corresponding w is one of those listed in lemmas 6,7 or 8. We now look more closely at such subgroups and show that in all cases other than the two required, the group $(P_\cap P^n)C_G(P_\cap P^n)$ has a normal p-subgroup which properly contains $P_\cap P^n$. The result then follows from three theorems of Brauer (1, 9F, 10B, 11B). As in section 2 it will be necessary to consider the different groups separately.

GL(m,q) , SL(m,q)

The Sylow p-subgroup of GL(m,q) is the group of upper-triangular matrices which have a 1 at each place on the main diagonal. Since this is also the Sylow p-subgroup of SL(m,q), we may consider the two groups together.

The correspondence between the generators of P and the previously mentioned root structure is as follows:-

$$x_{a_i-a_j}(u) \leftrightarrow I + uE_{ij}$$

where E_{ij} is the matrix whose only non-zero entry is a 1 in the (i,j)th place.

The relevant w are those of lemma 6.

If $w = \left(\begin{array}{c|c|c} 0 1 \ldots\ldots j & j+1 \ldots\ldots m-j-1 & m-j \ldots\ldots m \\ m\, m-1 \ldots\ldots m-j & j+1 \ldots\ldots m-j-1 & j \ldots\ldots 0 \end{array}\right)$

$P \cap P^n$ is the group of $(m+1)\times(m+1)$ matrices of the form

$$\begin{array}{cl}
\left[\begin{array}{ccccccccc}
1 & & & & & & & & \\
 & \ddots & & & & & 0 & & \\
 & & 1 & & & & & & \\ \hline
 & & & 1 & & * & & & \\
 & & & & \ddots & & & & \\
 & & & & & 1 & & & \\ \hline
 & & & & & & 1 & & \\
 & 0 & & & & & & \ddots & \\
 & & & & & & & & 1
\end{array}\right] &
\begin{array}{l} j+1 \\ \text{rows} \\ \\ \\ \\ \\ j+1 \\ \text{rows} \end{array}
\end{array}$$

To facilitate matters we first transform the elements of $P \cap P^n$ by the matrix

$$\begin{bmatrix} I_{j+1} & 0 & 0 \\ 0 & 0 & I_{m-2j-2} \\ 0 & I_{j+1} & 0 \end{bmatrix}$$

And so we may regard $P \cap P^n$ as the group of matrices of the form

$$\begin{array}{cl}
\left[\begin{array}{cccc|ccc}
1 & & & & & 0 & \\
 & \ddots & & & & & \\
 & & \ddots & & & & \\
 & & & 1 & & & \\ \hline
 & & & & 1 & & * \\
 & 0 & & & & \ddots & \\
 & & & & & & 1
\end{array}\right] &
\begin{array}{l} 2j+2 \\ \text{rows} \\ \\ \\ \\ \\ \\ \end{array}
\end{array}$$

Let g be any element of $GL(m+1,q)$

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A , B , C , and D are matrices of sizes $(2j+2)x(2j+2)$, $(2j+2)x(m-2j-1)$, $(m-2j-1)x(2j+2)$, and $(m-2j-1)x(m-2j-1)$ respectively.

Now suppose that g centralises all matrices of the form $\begin{bmatrix} I_{2j+2} & \\ & V \end{bmatrix}$, where V is any $(m-2j-1)x(m-2j-1)$ matrix of the form

$$\begin{bmatrix} 1 & & * \\ & \cdot & \\ & & 1 \end{bmatrix} .$$

The conditions for this are

$$B = BV$$

$$C = VC$$

$$DV = VD$$

for all V of the given type.

Thus
$$\begin{bmatrix} b_{11} & - - & b_{1s} \\ \vdots & & \vdots \\ b_{r1} & - - & b_{rs} \end{bmatrix} = \begin{bmatrix} b_{11} & \text{---} & b_{1s} \\ \vdots & & \vdots \\ b_{r1} & - - & b_{rs} \end{bmatrix} \begin{bmatrix} 1 & v_{12} & & v_{1s} \\ & \ddots & & \\ & & & v_{s-1\,s} \\ & & & 1 \end{bmatrix}$$

where v_{ij} , for $j > i$, is arbitrary

($r = 2j+2$, $s = m-2j-1$)

Consider the last column of the matrix on the right hand side of this equation, Equating this with the left hand side gives

$$b_{is} = b_{is} + \sum_{j=1}^{s-1} b_{ij} v_{js} \quad \text{for every } i = 1, \_\_\_, r$$

But the v_{ij} are arbitrary, and hence we may conclude that $b_{ij} = 0$ for every $j < s$.

A similar argument shows that $c_{ij} = 0$ for every $i > 1$.

Now consider the third of our conditions.

$$\begin{bmatrix} d_{11} & \cdots & d_{1s} \\ \vdots & & \vdots \\ d_{s1} & \cdots & d_{ss} \end{bmatrix} \begin{bmatrix} 1 & v_{12} & \cdots & v_{1s} \\ & \ddots & & \vdots \\ & & & v_{s-1\,s} \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & v_{1s} & \cdots & v_{1s} \\ & \ddots & & \vdots \\ & & & v_{s-1\,s} \\ & & & 1 \end{bmatrix} \begin{bmatrix} d_{11} & & d_{1s} \\ & & \\ d_{s1} & & d_{ss} \end{bmatrix}$$

where v_{ij} is once again arbitrary.

Equating (1,1)th. coefficients we find that

$$d_{11} = d_{11} + \sum_{j=2}^{s} v_{ij} \cdot d_{j1}$$

and hence that $d_{j1} = 0$ for every $j > 1$.

If we now apply a similar argument to the (2,2)th. coefficient we see that $d_{j2} = 0$ for every $j > 2$.

Proceeding in this way we eventually find that D is upper triangular.

Thus $d_{ij} = 0$ for $j < i$. Now consider the (1,j)th. coefficient on either side of the above equation, and suppose that all the v_{ik} are zero bar v_{ij}. We then have

$$d_{11} v_{ij} + d_{ij} = d_{ij} + v_{ij} d_{jj}$$

and hence that $d_{11} = d_{jj}$ for all j.

~~Interchanging the roles of g and g^{-1} we obtain~~
analogous results for C and F.
Combining (3.2d) with (3.1) we obtain as our final
condition

$$D\,(V-I)\,H \;=\; V-I$$

Similarly $H\,(V-I)\,D \;=\; V-I$

This implies in particular that D and H are upper-
~~triangular and have all diagonal terms equal.~~ Thus we
have the following lemma.

<u>Lemma 9</u> If $G = GL(m+1,q)$, and if $P_\cap P^n$ is the subgroup of G which consists of the matrices of the form $\begin{bmatrix} I_{2j+2} & \\ & V \end{bmatrix}$ the group $(P_\cap P^n).C_G(P_\cap P^n)$ consists of the matrices of the form

$$\left[\begin{array}{c|cc} A & 0 & \underline{b} \\ \hline \underline{c} & d & * \\ 0 & 0\ \ddots & d \end{array}\right]$$

where A is any non-singular $(2j+2)\times(2j+2)$ matrix, and $\underline{b}$, $\underline{c}$ are arbitrary column and row vectors, respectively, of length $2j+2$.

Denote the above group by Z, and let Q be the subgroup of Z which consists of those matrices for which $A = I$ and $d = 1$.

<u>Lemma 10</u> Q is a normal p-subgroup of Z which properly

contains $P_n P^n$.

Proof obvious

Clearly we can carry through a similar argument if we replace $GL(m+1,q)$ by $SL(m+1,q)$. Thus we have completed the proof of theorem 2 in these two cases.

Sp(2m,q)

The Sylow p-subgroup of $Sp(2m,q)$ is generated by elements of the form:-

$$ut_{ij} = \left[\begin{array}{c|c} I + uE_{ij} & \\ \hline & I - uE_{ji} \end{array}\right] \quad 1 \le i < j \le m$$

$$us_{ij} = \left[\begin{array}{c|c} I & u(E_{ij} + E_{ji}) \\ \hline & I \end{array}\right] \quad 1 \le i < j \le m$$

$$us_{ii} = \left[\begin{array}{c|c} I & uE_{ii} \\ \hline & I \end{array}\right] \quad 1 \le i \le m$$

where I is the unit mxm matrix, u belongs to $GF(q)$, and E_{ij} is the mxm matrix whose only non-zero entry is a 1 in the (i,j)th place.

The correspondence between these generators and the previously mentioned root structure is as follows:-

$$x_{a_i - a_j}(u) \longleftrightarrow ut_{ij}$$

$$x_{a_i + a_j}(u) \longleftrightarrow us_{ij}$$

The relevant w are those of lemma 7, and it will be recalled that these fell into two classes.

<u>Case 1</u> there exists an s in the range 1 to m+1 such that $w(a_t) = a_t$ for $t \geq s$, $w(a_u) = -a_u$ for $u < s$.

Then $P \cap P^n = Gp.\left\{ x_r(t) \;;\; \begin{array}{ll} r = a_i + a_j & i,j \geq s \\ \text{or } r = a_i - a_j & j > i \geq s \end{array} \right\}$

$C_{Sp(2m,q)}(P \cap P^n) = Sp(2m,q) \cap C_{GL(2m,q)}(P \cap P^n)$, and so it will be more convenient if we first consider $C_{GL(2m,q)}(P \cap P^n)$.

Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a matrix which belongs to this latter group. (A, B, C, D are all mxm matrices.) Then we have two centraliser conditions, the first of which is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & S \\ & I \end{bmatrix} = \begin{bmatrix} I & S \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

i.e. $$\begin{bmatrix} A & AS+B \\ C & CS+D \end{bmatrix} = \begin{bmatrix} A+SC & B+SD \\ C & D \end{bmatrix}$$

where S is any symmetric mxm matrix, the first s-1 rows and columns of which are zero.

Our condition reduces to
$$CS = SC = 0$$
$$AS = SD$$

Hence we deduce that B may be arbitrary, and if we put the non-zero minor of S equal to I_{m-s+1} we see that

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$ where C_1 is any $(s-1)x(s-1)$ matrix

Similarly if we write A and D in the form $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, $\begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$

where A_1 and D_1 are $(s-1)x(s-1)$ matrices, we see that $A_2 = 0$, $D_3 = 0$, and $A_4 = D_4$. A_4 must also satisfy the condition $\bar{S}A_4 = A_4\bar{S}$ for every symmetric $(m-s+1)x(m-s+1)$ matrix $\bar{S}$.

Our other centraliser condition is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix} = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $T_1 = I + V$, $T_2 = I + V'$ (V' is the transpose of V) and V is an upper-triangular matrix mxm matrix having zeros along the main diagonal and at all places in the first s-1 rows.

Therefore $$\begin{bmatrix} AT_1 & BT_2 \\ CT_1 & DT_2 \end{bmatrix} = \begin{bmatrix} T_1A & T_1B \\ T_2C & T_2D \end{bmatrix}$$

As with the case of $GL(m,q)$, $AT_1 = T_1A$ implies that

$$A = \left[\begin{array}{c|ccc} A_1 & & 0 \; \underline{b} & \\ \hline \underline{c} & \lambda & 0 & \mu \\ & & \ddots & 0 \\ 0 & 0 & & \lambda \end{array}\right]$$

Combining this with our earlier conditions we see that

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline \underline{c} & \\ 0 & \lambda I \end{array}\right]. \text{ Similarly } D = \left[\begin{array}{c|c} D_1 \ \underline{f} & 0 \\ \hline 0 & \lambda I \end{array}\right]$$

N.B. λ is the same for both A and D.

The condition $CT_1 = T_2C$ is automatically satisfied for a C of the form given by the earlier condition.

If we write B in the form $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$ where B_1 is an $(s-1)x(s-1)$ matrix, the condition $BT_2 = T_1B$ implies that $B_2V' = 0$, $VB_3 = 0$, and $VB_4 = -B_4V'$.

Whence $B_2 = \begin{bmatrix} \underline{g} & 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} \underline{h} \\ 0 \end{bmatrix}$

where $\underline{g}$, $\underline{h}$ are column and row vectors, respectively, each of length s-1.

If E_{ij} is the $(m-s+1)x(m-s+1)$ matrix whose only non-zero entry is a 1 in the (i,j)th place, the condition $VB_4 = -B_4V'$ is equivalent to $E_{ij}B_4 = -B_4E_{ji}$ for all i,j such that $1 \leq i < j \leq m-s+1$.

Letting $B_4 = (b_{ij})$ we have

$$E_{ij}B_4 = \begin{bmatrix} 0 \\ b_{j1} \ b_{j2} \cdots\cdots \\ 0 \end{bmatrix} \text{— ith row}$$

$$-B_4 E_{ji} = \begin{array}{c} \left[\begin{matrix} & -b_{1j} & \\ & -b_{2j} & \\ 0 & \cdot & 0 \\ & \cdot & \\ & & \\ & \cdot & \end{matrix}\right] \\ \quad i\text{th. col.} \end{array}$$

Hence our condition implies that (i) $b_{jk} = 0$ if $k \neq i$ (ii) $b_{kj} = 0$ if $k \neq i$ (iii) $b_{ij} = -b_{ji}$.

Fix j. Then i can be any integer in the range 1 to j-1. Thus, if there exists more than one such i we can conclude that $b_{jk} = b_{kj} = 0$ for every k.

Hence we have $$B_4 = \left[\begin{matrix} y & z & 0 & \cdots & & 0 \\ -z & 0 & \cdots & & & 0 \\ 0 & \cdot & & & & \\ \cdot & \cdot & 0 & & 0 & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & 0 & & 0 & \\ 0 & 0 & & & & \end{matrix}\right]$$

where y,z are any elements of GF(q).

Thus we have

<u>Lemma 11</u> If there exists an s such that $w(a_t) = a_t$ for $t \geq s$, while $w(a_u) = -a_u$ for $u < s$, $C_{Sp(2m,q)}(B_n P^n)$ is the set of matrices in Sp(2m,q) which have the form

$$\left[\begin{array}{c|c|c|c} A_1 & 0 & B_1 & \underline{g} \quad 0 \\ \hline \underline{c} & & \underline{h} & \begin{matrix} y & z & 0 & \cdots & 0 \\ -z & 0 & \cdots & & 0 \end{matrix} \\ & \lambda I & & \\ 0 & & 0 & 0 \\ \hline C_1 & 0 & D_1 & \underline{f} \quad 0 \\ \hline 0 & 0 & 0 & \lambda I \end{array}\right]$$

where A_1, B_1, C_1, D_1 are arbitrary (s-1)x(s-1) matrices; $\underline{c}$, $\underline{f}$, $\underline{g}$, $\underline{h}$ are arbitrary (s-1)-vectors; y,z are any elements of GF(q); and λ is any element of GF(q)*. Hence $(P \cap P^n).C_{Sp(2m,q)}(P \cap P^n)$ is the subgroup of Sp(2m,q) which consists of those matrices of the form

$$\left[\begin{array}{c|c|c|c} A_1 & 0 & B_1\ \underline{g} & 0 \\ \hline \underline{c} & \lambda(I+V) & \underline{h} & Z \\ \hline C_1 & 0 & D_1\ \underline{f} & 0 \\ \hline 0 & 0 & 0 & \lambda(I-V') \end{array}\right]$$

where V is upper-triangular with zeros on the main diagonal, and Z has the form $(I+V)S + \begin{bmatrix} y & z & 0 & \cdots & 0 \\ -z & 0 & \cdots & & 0 \\ 0 & 0 & & & \\ \cdot & \cdot & & 0 & 0 \\ 0 & 0 & & & \end{bmatrix}$

and S is symmetric.

Denote this group by R, and let Q be the subset of R which consists of those matrices for which $A_1 = D_1 = I$; $B_1 = C_1 = 0$; $\lambda = 1$, $y = z = 0$.

<u>Lemma 12</u> Q is a normal p-subgroup of R which properly contains $P \cap P^n$.

<u>Proof</u> To show that it is a subgroup it is only necessary to see that if S,V,V' are as described there exists a symmetric matrix S_1 such that $S(I-V') = (I+V)S_1$.

Since the subgroup of $GL(2m,q)$ which consists of matrices of this form is a p-group , it follows that Q is a p-group.

The normality of Q is easy to check.

To show that $P \cap P^n$ is a proper subgroup it suffices to produce an element of Q which does not belong to $P \cap P^n$. Such an element is obtained by putting $S = 0$, $\underline{c} = \underline{f} = 0$ $\underline{g} = \underline{h} \neq 0$.

<u>Case 2</u> The prospective defect group of this case is simply that of case 1 augmented by the extra generators x (t) , for u , and t $GF(q)$.

Thus the"defect group's" centraliser in this case is a subgroup of that of case 1. This means that we simply have extra conditions on A , B , etc.

If we let $A = (a\ )$, $B = (b\ )$ etc. , and $\underline{g} = (g\ ,\ldots\ldots,g\ )$, $\underline{h} = (h\ ,\ldots\ldots,h\ )$ etc. , it is merely a matter of matrix multiplication to verify that the required conditions are

$a = 0$ for every $j = u$, $a = 0$ for every $i = u+1$,

and $a = a$.

$b = 0$ for every $j = u$, $b = 0$ for every $i = u$,

and $-b = b$.

$c = 0$ for every $j = u+1$, $c = 0$ for every $i = u+1$,

and $-c = c$.

$d_{j,u+1} = 0$ for every $j \neq u+1$, $d_{u,i} = 0$ for every $i \neq u$,

and $d_{u,u} = d_{u+1,u+1}$.

$c_u = f_u = h_{u+1} = g_{u+1} = 0$.

We have these conditions for each $u \in \mathcal{X}_w$.

Let D_1 be the " $P_n P^n$ " of case 1 and that of case 2 be D_2.

Let K_1 be the centraliser in Sp.(2m,q) of D_1 and K_2 be that of D_2 .

Then , as we remarked earlier , D_1 is a subgroup of D_2, while K_2 is a subgroup of K_1 .

In lemma 11 we showed that the group Q , described in case 1 , is a normal p-subgroup of D_1K_1 .

Therefore Q is normalised by K_1 , and hence by K_2.

Therefore $Q \cap K_2$ is a normal subgroup of K_2.

Since Q is a p-group , and $Q \cap K_2$ centralises D_2, it now follows that $(Q \cap K_2).D_2$ is a normal p-subgroup of K_2D_2 .

As we remarked on page 16 , no two consecutive indices can belong to $\mathcal{X}_w$ (or $\mathcal{Y}_w$) . Thus the above conditions on $\underline{h}$, $\underline{g}$, etc. do not demand that $\underline{h}$, $\underline{g}$ be zero. Hence as in case 1 , we can produce an element of $Q \cap K_2$ which does not lie in D_2 . Therefore D_2 is a proper subgroup of $(Q \cap K_2).D_2$. This completes the proof of theorem 2 for Sp(2m,q) .

~~of $(P_\cap P^n)$. $C_{Sp(2m,q)}(P_\cap P^n)$ which properly contains $P_\cap P^n$.~~

~~This completes the proof of theorem 2 for Sp(2m,q).~~

<u>$\Omega(2m,q)$</u>

The Sylow p-subgroup of $\Omega(2m,q)$ is generated by elements of the form:-

$$ut_{ij} = \left[\begin{array}{c|c} I + uE_{ij} & \\ \hline & I - uE_{ji} \end{array}\right] \qquad 1 \leq i < j \leq m$$

$$us_{ij} = \left[\begin{array}{c|c} I & u(E_{ij} - E_{ji}) \\ \hline & I \end{array}\right] \qquad 1 \leq i < j \leq m$$

where I is the unit mxm matrix, u belongs to GF(q), and E_{ij} is the mxm matrix whose only non-zero entry is a 1 in the (i,j)th place.

The correspondence between these generators and the previously mentioned root structure is as follows:-

$$x_{a_i - a_j}(u) \leftrightarrow ut_{ij}$$
$$x_{a_i + a_j}(u) \leftrightarrow us_{ij}$$

The relevant w are those of lemma 8, which, it will be remembered, came in four types. Fortunately the first two are very similar to the analogous cases for Sp(2m,q). As before we shall first consider $C_{GL(2m,R)}(P_\cap P^n)$.

<u>Case 1</u> there exists an s in the range 1 to m+1 such

that $w(a_t) = a_t$ for $t \geqslant s$, while $w(a_u) = -a_u$ for $u < s$.

Then $P_\wedge P^n = \mathrm{Gp.}\left\{ x_r(t) \;;\; \begin{array}{l} r = a_i + a_j \text{ for } s \leq i < j \\ \text{or } r = a_i - a_j \text{ for } s \leq i < j \end{array} \right\}$

Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C, D are $m \times m$ matrices, be any element of $C_{GL(2m,q)}(P_\wedge P^n)$. Then we have two centraliser conditions, the first of which says that for every $m \times m$ skew-symmetric matrix of the form

$$S = \begin{bmatrix} 0 & 0 \\ 0 & \bar{S} \end{bmatrix}$$

where $\bar{S}$ is an $m-s+1 \times m-s+1$ matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & S \\ & I \end{bmatrix} = \begin{bmatrix} I & S \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

i.e.
$$CS = SC = 0$$
$$AS = SD$$

If we write C in the form $\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$ where C_1 is an $(s-1)\times(s-1)$ matrix we see that

$$C_2 \bar{S} = 0, \quad \bar{S} C_3 = 0, \quad C_4 \bar{S} = \bar{S} C_4 = 0$$

Hence we can conclude that $C_2 = 0$, $C_3 = 0$, $C_4 = 0$.

If we write A and D in a similar form we obtain the condition

$$AS = \begin{bmatrix} 0 & A_2\bar{S} \\ 0 & A_4\bar{S} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \bar{S}D_3 & \bar{S}D_4 \end{bmatrix} = SD$$

whence we conclude that $A_2 = 0$, $D_3 = 0$, and $A_4\bar{S} = \bar{S}D_4$.
The other centraliser condition, involving the ut_{ij}, is precisely the same as for $Sp(2m,q)$.

Hence we obtain the following lemma:-

<u>Lemma 13</u> If there exists an s such that $w(a_t) = a_t$ for $t \geqslant s$, while $w(a_u) = -a_u$ for $u < s$, $C_{\Omega(2m,q)}(\mathfrak{P} \cap P^n)$ is the set of matrices in $\Omega(2m,q)$ which have the form

$$\left[\begin{array}{c|c|c|ccc}
A_1 & 0 & B_1 & \underline{g} & & 0 \\ \hline
\underline{c} & & \underline{h} & y & z\ 0 \cdots\cdots & 0 \\
 & & & -z & 0 \cdots\cdots\cdots & 0 \\
0 & \lambda I & & 0 & \cdot & \\
 & & & \cdot & & \\ \hline
C_1 & 0 & D_1 & \underline{f} & & 0 \\ \hline
0 & 0 & 0 & & \lambda I &
\end{array}\right]$$

where A_1, B_1, C_1, D_1 are arbitrary $(s-1)\mathrm{x}(s-1)$ matrices; $\underline{c}$, $\underline{f}$, $\underline{g}$, $\underline{h}$ are arbitrary $(s-1)$-vectors; y, z are any elements of $GF(q)$; and λ is any element of $GF(q)^*$.

Let Q be the set of matrices in $\Omega(2m,q)$ which have the form

$$\left[\begin{array}{c|c|c|cc}
I & 0 & 0 & \underline{g} & 0 \\ \hline
\underline{c} & & \underline{h} & & \\
0 & I+V & 0 & & Y \\ \hline
0 & 0 & I & \underline{f} & 0 \\ \hline
0 & 0 & 0 & & I-V
\end{array}\right]$$

where $Y = (I+V)S + \begin{bmatrix} y & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \cdot & & 0 & \\ 0 & & & \end{bmatrix}$

V is an upper triangular matrix having zeros down the main diagonal; V' is the transpose of V; S is a skew-symmetric matrix; $\underline{c}$, $\underline{h}$, $\underline{g}$, $\underline{f}$ are row and column vectors of length $s-1$; and y is any element of $GF(q)$.

Lemma 14. Q is a normal p-subgroup of $(P_\cap P^n).C_{\Omega(2m,q)}(P_\cap P^n)$ and it properly contains $P_\cap P^n$.

Proof The easiest way to show that Q is a p-group is to show that the correspond~~ing~~ set of matrices in $GL(2m,q)$ is a p-group. Showing that this latter set is a group is simply a matter of matrix multiplication.

Let X be a matrix of the given form. X^p is a matrix for which the $\underline{c}$, $\underline{g}$ etc. terms have vanished. $(I+V)$ is a matrix whose order is a power of p, say p^α.

Therefore X^{p^α} belongs to the Sylow p-subgroup of $GL(2m,q)$

Hence Q is a p-group.

To show that Q is normal, it suffices to show that the corresponding group in $GL(2m,q)$ is normal in $(P_\cap P^n).C_{GL(2m,q)}(P_\cap P^n)$. This again is simply a matter of matrix multiplication.

The element obtained by putting $\underline{c} = 0$, $\underline{f} = 0$, $V = 0$, $S = 0$, $y = 0$, $\underline{g} = -\underline{h}' \neq 0$ does not belong to $P_\cap P^n$, and yet does belong to $\Omega(2m,q)$, since it may be expressed in terms of the us_{ij}. Hence Q properly contains $(P \cap P^n)$, which completes the proof of lemma 14.

Case 2 there exists an s in the range 3 to m+1 such that $w(a_t) = a_t$ for $t \geq s$, $w(a_u) = -a_{u+1}$ for $u \in X_w$, $w(a_u) = -a_{u-1}$ for $u \in Y_w$ while $w(a_u) = -a_u$ for $u \in Z_w$

This is analogous to case 2 of Sp(2m,q). Similar modifications apply, and so we shall not set down the details.

Case 3 $w(a_m) = a_{m-1}$, $w(a_{m-1}) = a_m$, $w(a_i) = -a_i$ for all other i.

$$P \cap P^n = Gp\{x_{a_m+a_{m-1}}(t) \ ; \ t \text{ belongs to } GF(q)\}$$

In matrix terms this is the group of matrices of the form

$$\left[\begin{array}{c|c} I & \begin{matrix} & & 0 & 0 \\ 0 & & \cdot & \cdot \\ & & \cdot & 0 \\ & & 0 & y \\ 0 \ldots 0 & & -y & 0 \end{matrix} \\ \hline & I \end{array}\right]$$

where y is any element of GF(q).

As before the condition for $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to centralise this group is $CS = SC = 0$, $AS = SD$ where this time S is any mxm matrix of the form

$$\begin{bmatrix} & & & 0 & 0 \\ 0 & 0 & & \cdot & \cdot \\ & & & \cdot & 0 \\ 0 & \ldots\ldots & & 0 & y \\ 0 & \ldots & & 0 & -y & 0 \end{bmatrix}$$

This implies that A, C, D have the forms given below

$$A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline A_3 & \begin{matrix} d_1 & d_2 \\ d_3 & d_4 \end{matrix} \end{array}\right], \qquad D = \left[\begin{array}{c|c} D_1 & D_2 \\ \hline 0 & \begin{matrix} d_4 & -d_3 \\ -d_2 & d_1 \end{matrix} \end{array}\right]$$

$$C = \left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where A_1, C_1, D_1, are $(m-2)x(m-2)$ matrices; A_3 is an $2x(m-2)$ matrix, D_2 is an $(m-2)x2$ matrix; and d_1, d_2, d_3, d_4 are elements of $GF(q)$.

Let Q be the set of matrices which (i) belong to $\Omega(2m,q)$ and (ii) have the form

$$\left[\begin{array}{c|c|c|c} I & 0 & 0 & G_2 \\ \hline F & I & G_3 & G_4 \\ \hline 0 & 0 & I & H \\ \hline 0 & 0 & 0 & I \end{array}\right]$$

where F, G_3 are $2x(m-2)$ matrices; G_2, H are $(m-2)x2$ matrices; and G_4 is a $2x2$ matrix.

Lemma 15 Q is a normal p-subgroup of $(P \cap P^n).C_{\Omega(2m,q)}(P \cap P^n)$ which properly contains $P \cap P^n$.

Proof The proof that Q is a normal p-subgroup is essentially the same as that of lemma 14.

The element which shows that the inclusion is proper is obtained by putting $F = 0$, $H = 0$, $G_4 = 0$, and $G_3 = -G_2'$.

Case 4 $w(a_m) = a_{m-1}$, $w(a_{m-1}) = a_m$, $w(a_u) = -a_{u+1}$ for $u \in \mathcal{X}_w$, $w(a_u) = -a_{u-1}$ for $u \in \mathcal{Y}_w$, and $w(a_u) = -a_u$ for $u \in \mathcal{Z}_w$.

The $P_\cap P^n$ of this case is that of case 3 augmented by the extra generators $x_{a_u - a_{u+1}}(t)$ for $u \in \mathcal{X}_w$. Thus the centraliser in this case is a subgroup of that of case 3.

If we denote this new centraliser by K , the new $P_\cap P^n$ by D , and let Q be the group described in lemma 15 , we show by an argument analogous to that of page 32 that $Q_\cap K$ is a normal p-subgroup of DK . And so if we can show that $Q_\cap K$ contains an element which is not in D , we shall have the by now familiar situation of a normal p-subgroup of KD properly containing D .

It is sufficient , therefore , to consider the effect on Q of the new centraliser conditions. Finding these new restrictions on the coefficients of the matrices in Q is again a matter of matrix multiplication. They are similar to those of case 2 of Sp(2m,q) , and the important thing to note is that they do not force G_2 and G_3 to be zero. This means that, as in case 3, we can produce elements of $Q_\cap K$ which do not lie in D .

By this time it will come as no surprise that Q is a normal p-subgroup of $(P \cap P^n).C_{\Omega(2m,q)}(P \cap P^n)$, and that it properly contains $P \cap P^n$.

The proof of this statement is again that of lemma 14. The element which lies in Q but not in $P \cap P^n$ is obtained by putting $\underline{a} = 0$, $\underline{d} = 0$, $g = h = v = 0$, and $\underline{b} = -\underline{c}'$.

This completes the proof of theorem 2 for $\Omega(2m,q)$, and it will doubtless hearten any reader that I might still have to realise that this leaves but one more group to be considered.

$\underline{\Omega(2m+1,q)}$

The Sylow p-subgroup of $\Omega(2m+1,q)$ is generated by elements of the form:-

$$ut_{ij} = \begin{bmatrix} 1 & & \\ & I + uE_{ij} & \\ & & I - uE_{ji} \end{bmatrix} \qquad 1 \leq i < j \leq m$$

$$us_{ij} = \begin{bmatrix} 1 & & \\ & I & u(E_{ij} - E_{ji}) \\ & & I \end{bmatrix} \qquad 1 \leq i < j \leq m$$

$$\begin{array}{ccc} & & \text{col.}\, i \\ ur_i = & \begin{bmatrix} 1 & & u \\ -2u & I & -u^2 E_{ii} \\ & & I \end{bmatrix} & \\ \text{row } i & & \end{array} \qquad 1 \leq i \leq m$$

where I is the unit $m \times m$ matrix, u belongs to $GF(q)$, and E_{ij} is the $m \times m$ matrix whose only non-zero entry is a 1 in the (i,j)th place.

The correspondence between these generators and the previously mentioned root system is as follows:-

$$x_{a_i - a_j}(u) \longleftrightarrow u t_{ij}$$
$$x_{a_i + a_j}(u) \longleftrightarrow u s_{ij}$$
$$x_{a_i}(u) \longleftrightarrow u r_i$$

The relevant w are those of lemma 7, which means that there are two cases to be considered.

<u>Case 1</u> there exists an s in the range 1 to m+1 such that $w(a_t) = a_t$ for $t \geq s$, $w(a_u) = -a_u$ for $u < s$.

Then $$P \cap P^n = \mathrm{Gp.}\left\{ x_r(t) \; ; \; \begin{array}{l} r = a_i + a_j \text{ for } s \leq i < j \\ \text{or } r = a_i - a_j \text{ for } s \leq i < j \\ \text{or } r = a_i \text{ for } s \leq i \end{array} \right\}$$

Suppose that $$\left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right]$$

where A , B , C , D are $m \times m$ matrices, is any element of $C_{GL(2m+1,q)}(P \cap P^n)$.

We have three centraliser conditions, the first of which is

$$\left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right] \left[\begin{array}{c|c|c} 1 & & \\ \hline & I & S \\ \hline & & I \end{array}\right] = \left[\begin{array}{c|c|c} 1 & & \\ \hline & I & S \\ \hline & & I \end{array}\right] \left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right]$$

for every skew-symmetric matrix S whose first s-1 rows are zero.

Therefore $SC = CS = 0$, $AS = SD$, $\underline{a}\,S = 0$, and $S\,\underline{d} = 0$.

Our second condition is

$$\left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right] \left[\begin{array}{c|c|c} 1 & & \\ \hline & T_1 & \\ \hline & & T_2 \end{array}\right] = \left[\begin{array}{c|c|c} 1 & & \\ \hline & T_1 & \\ \hline & & T_2 \end{array}\right] \left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right]$$

where $T_1 = I+V$, $T_2 = I-V'$ and V is an upper-triangular mxm matrix having zeros along the main diagonal and at all places in the first s-1 rows.

This is equivalent to

$$AT_1 = T_1 A \quad , \quad BT_2 = T_1 B \quad , \quad CT_1 = T_2 C$$
$$DT_2 = T_2 D \quad , \quad \underline{a}\,T_1 = \underline{a} \quad , \quad \underline{b}\,T_2 = \underline{b}$$
$$\underline{c} = T_1\underline{c} \quad , \quad \underline{d} = T_2\underline{d}$$

The third condition is

$$\left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right] \left[\begin{array}{c|c|c} 1 & & u \\ \hline -2u & I & -u^2 E_{ii} \\ \hline & & I \end{array}\right] = \left[\begin{array}{c|c|c} 1 & & u \\ \hline -2u & I & -u^2 E_{ii} \\ \hline & & I \end{array}\right] \left[\begin{array}{c|c|c} z & \underline{a} & \underline{b} \\ \hline \underline{c} & A & B \\ \hline \underline{d} & C & D \end{array}\right]$$

for every u belonging to $GF(q)$ and every $i \geqslant s$.

The net result of all this is the following lemma:-

<u>Lemma 16</u> If there exists an s in the range 1 to $m+1$ such that $w(a_t) = a_t$ for $t \geqslant s$, while $w(a_u) = -a_u$ for $u < s$ $C_{\Omega(2m+1,q)}(P \cap P^n)$ is the group of matrices which (i) belong to $\Omega(2m+1,q)$ and (ii) have the form

$$\left[\begin{array}{c|c|c|c|c}
\lambda & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\
\hline
 & A_1 & 0 & B_1 & \underline{g} \quad 0 \\
\cline{2-5}
0 & \underline{c} & & \underline{h} & y \quad z \quad 0 \cdots 0 \\
 & & & & -z \quad 0 \cdots\cdots 0 \\
 & 0 & \lambda I & 0 & 0 \\
\cline{2-5}
 & C_1 & 0 & D_1 & \underline{f} \quad 0 \\
\cline{2-5}
0 & 0 & 0 & 0 & \lambda I
\end{array}\right]$$

where A_1, B_1, C_1, D_1 are $(s-1)\times(s-1)$ matrices; $\underline{c}$, $\underline{g}$, $\underline{h}$, $\underline{f}$ are row and column vectors of length $s-1$; y, z are elements of $GF(q)$; and λ is an element of $GF(q)^*$.

$P \cap P^n$ is the group of matrices of the form

$$\left[\begin{array}{c|c|c|c|c}
1 & \multicolumn{2}{c|}{0} & 0 & \underline{r}\,(I-V') \\
\hline
0 & I & 0 & 0 & 0 \\
\hline
-2\underline{r}' & 0 & I+V & 0 & (R+S)(I-V') \\
\hline
 & 0 & 0 & I & 0 \\
\cline{2-5}
0 & 0 & 0 & 0 & I-V'
\end{array}\right]$$

where R is a matrix of the form

$$\begin{bmatrix} y_s^2 & -2y_s y_{s+1} & \cdots\cdots & -2y_s y_m \\ & y_{s+1}^2 & & \vdots \\ & & \ddots & \vdots \\ & & & -2y_{m-1} y_m \\ & & & y_m^2 \end{bmatrix}$$

$\underline{r}$ the vector $(y_s , y_{s+1} , \ldots\ldots\ldots y_m)$; V is an upper-triangular matrix with zeros down the main diagonal ; and S is skew-symmetric.

From this it is a straightforward matter to write down the form of the matrices in $(P_\cap P^n).C_{\Omega(2m+1,q)}(P_\cap P^n)$.

Let Q be the set of matrices in $\Omega(2m+1,q)$ which have the form

$$\left[\begin{array}{c|c|c|c|cc} 1 & \multicolumn{2}{c|}{0} & 0 & \multicolumn{2}{c}{\underline{r}(I-V')} \\ \hline 0 & I & 0 & 0 & \underline{g} & 0 \\ -2\underline{r}' & \begin{matrix}\underline{c}\\ 0\end{matrix} & I+V & \begin{matrix}\underline{h}\\ 0\end{matrix} & \multicolumn{2}{c}{Y} \\ \hline 0 & 0 & 0 & I & \underline{f} & 0 \\ \hline 0 & 0 & 0 & 0 & \multicolumn{2}{c}{I-V'} \end{array}\right]$$

where Y is a matrix of the form

$$(R+S)(I-V') + \begin{bmatrix} z & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}$$

z being any element of GF(q).

AS previously we claim that Q is a normal p-subgroup of $(P \cap P^n).C_{\Omega(2m+1,q)}(P \cap P^n)$, and that it properly contains $P \cap P^n$.

The proof is essentially that of lemma 14.

<u>Case 2</u> there exists an s in the range 3 to $m+1$ such that $w(a_t) = a_t$ for $t \geqslant s$, $w(a_u) = -a_{u+1}$ for $u \in \mathcal{X}_w$, $w(a_u) = -a_{u-1}$ for $u \in \mathcal{Y}_w$ while $w(a_u) = -a_u$ for $u \in \mathcal{Z}_w$

$P \cap P^n$ is the group of case 1 augmented by the extra generators $x_{a_u - a_{u+1}}(t)$, where t belongs to $GF(q)$, $u \in \mathcal{X}_w$

As will readily be seen this is analogous to case 2 of $Sp(2m,q)$, and similar modifications yield the required result.

This completes the proof of theorem 2.

<u>§4</u>

Theorem 2 shows that, if G is a group of the type described, the blocks of G are all of either highest or lowest kind. We have still to show that both types do in fact occur, and that the theorem also holds for the non-exceptional finite Chevalley groups. We also need to find the block idempotents in the various cases. In this section we shall complete these tasks.

The first problem is resolved fairly easily. The centres of the groups $GL(m,q)$, $SL(m,q)$, $Sp(2m,q)$, $\Omega(2m,q)$ and

$\Omega(2m+1,q)$ all have order not divisible by p. Hence in all cases we can form a central idempotent, E, of kG as follows:-

$$E = \frac{1}{\text{order of centre}} \cdot \sum(\text{all elements in the centre})$$

Therefore kGE is a sum of blocks of kG.

But $kGE \cong k\left(G/_{\mathfrak{Z}(G)}\right)$ as k(GxG)-modules, and so if $G/_{\mathfrak{Z}(G)}$ were to have a block of neither highest nor lowest kind so also would G. Hence theorem two is true for the groups $PGL(m,q)$, $PSL(m,q)$, $PSp(2m,q)$, $P\Omega(2m,q)$, $P\Omega(2m+1,q)$.

As we mentioned in our introduction Steinberg (14) proved that the latter four classes of groups all have a block of defect zero. He had earlier (13) proved a similar theorem for $PGL(m,q)$. Hence all ten classes of groups have blocks of highest and lowest kind, and no other type of block is possible.

In order to find the frequency and idempotents of the various blocks we shall, of course, have to consider the groups individually.

<u>GL(m,q)</u>

If we define the defect group of a conjugacy class to be the Sylow p-subgroup of the centraliser of a class representative, there is a theorem of Brauer which states that the number of blocks of maximal defect is equal to the number of p-regular classes of maximal defect.

There are a number of proofs of this theorem; a good one is to found in (12).

Class representatives for the p-regular classes are matrices of the form

$$\begin{bmatrix} y_1 & & & & \\ & y_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & y_m \end{bmatrix}$$

And so we have to find those matrices of this type which centralise P.

The condition for this is

$$y_i p_{ij} y_j^{-1} = p_{ij} \quad \text{for every } p_{ij} \text{ belonging to } GF(q) \text{ and every } i, j.$$

$$\text{i,e.} \qquad y_i = y_j \quad \text{for every } i, j$$

The matrices λI are all in different conjugacy classes and so we conclude that $GL(m,q)$ has $q-1$ blocks of maximal defect. Steinberg (13) shows that this is also the number of blocks of defect zero.

The centre of $GL(m,q)$ is cyclic of order $q-1$. Let it be generated by ρ and let ω be a primitive $(q-1)$st root of unity.

Let $\quad E_i = \dfrac{1}{q-1} \sum_{r=1}^{q-1} \omega^{ir} \rho^r \quad$ for $i = 0, 1, 2, \ldots\ldots q-2$

Then the E_i are central idempotents of kG, and

$$kG \cong \bigoplus_{i=0}^{q-2} kGE_i \qquad \text{as } k(G\times G)\text{-modules.}$$

$$\dim_k kGE_i = q^{\frac{m(m-1)}{2}}.(q^m-1).(q^{m-1}-1)\ldots\ldots(q^2-1)$$

The dimension of each block of defect zero is $q^{\frac{m(m-1)}{2}}$, and since there are only q-1 of these, each summand kGE_i must contain one, and hence only one, block of maximal defect. Thus to find the block idempotents it is sufficient to determine the distribution of the blocks of defect zero among the kGE_i.

Let $\mathcal{E}$ be a primitive (q-1)st root of unity and $\hat{\chi}$ a (modular) Steinberg character i.e. a character of defect zero.

Then Steinberg shows in (13) that

$$\hat{\chi}(\rho) = q^{\frac{m(m-1)}{2}} \mathcal{E}^{mu}$$

where u is some integer between 1 and q-1, the q-1 values of u corresponding to the q-1 Steinberg characters.

As is well known, if φ is a character of a group G and e a block idempotent of G, $\varphi(e) = 0$ unless e is the block to which φ belongs. In this latter case $\varphi(e) = \varphi(1)$.

Hence to discover to which E_i we may assign $\hat{\chi}$, it is sufficient to find that E_i for which $\hat{\chi}(E_i) \neq 0$.

$$\hat{\chi}(E_i) = \frac{q^{\frac{m(m-1)}{2}}}{q-1} \sum_{r=1}^{q-1} \bar{\omega}^{ir} \mathcal{E}^{mur}$$

which is non-zero iff $\bar{\omega}^i = \mathcal{E}^{-mu}$.

$\bar{\omega}$ being the complex (q-1)st root of unity which corresponds to ω under the Brauer character construction

Without loss of generality we may assume that $\bar{\omega} = \varepsilon$ and hence the number of blocks of defect zero which are contained in the summand kGE_i is equal to the number of solutions of $i + mu \equiv 0\ (q-1)$ which lie in the range 1 to $q-1$.

Let d be the h.c.f. of m and $q-1$.

Then $q-1 = xd$, $m = yd$ whence we have that u is a solution iff xd divides $i+ydu$.

This can only occur if d divides i.

If d does divide i, then the number of solutions in the given range is readily seen to be d.

In particular if $i = 0$ the number of blocks of defect zero attached to E_i is d. But we already know that

$$kGE = k\left(G/_{\mathfrak{J}(G)}\right)$$

and so we can deduce that $PGL(m,q)$ has d blocks of defect zero.

Thus we have proved

<u>Theorem 3</u> $GL(m,q)$ has $q-1$ blocks of defect zero and $q-1$ of maximal defect. $PGL(m,q)$ has one block of maximal defect and $d = (m,q-1)$ of defect zero.

The block idempotents of $GL(m,q)$ are as follows.

Let $\chi_1, \chi_2, \ldots\ldots, \chi_{q-1}$ be the Steinberg characters as given in (13), and let

$$e_j = \frac{1}{|GL(m,q)|} \chi_j(1) \sum_{\alpha} \chi_j(g_\alpha^{-1}) K_\alpha$$

where the summation is over the conjugacy classes of G, g_α is any element of the class R_α, and K_α is the class sum of R_α.

Then, as is well known, $e_1, \ldots\ldots e_{q-1}$ are the block idempotents for the blocks of defect zero.

Let $E_i' = E_i - \sum e_j$, where the summation is over all j such that $\chi_j(E_i) \neq 0$

Then $E_1', \ldots\ldots, E_{q-1}'$ are the idempotents for the blocks of maximal defect.

The idempotents for PGL(m,q) are calculated in a similar manner.

<u>SL(m,q)</u>

As with GL(m,q) class representatives for the p-regular classes are provided by the scalar matrices λI. The number of such matrices is the number of elements of GF(q) which satisfy the equation $\lambda^m = 1$. Hence Brauer's criterion tells us that the number of blocks of maximal defect is d , the highest common factor of m and q-1 .

The centre of SL(m,q) is cyclic of order d. We again suppose that it is generated by Υ and form central idempotents

$$E_i = \frac{1}{d}\sum_{r=1}^{d} \omega^{ir} \rho^r$$

where this time ω is a primitive dth root of unity. Since $SL(m,q)$ is a normal subgroup of $GL(m,q)$ we can apply Clifford's theorem, i.e. if χ is an irreducible character of $GL(m,q)$, $\chi_{SL(m,q)}$ is a sum of irreducible conjugate characters of $SL(m,q)$. Thus the only irreducible characters of $GL(m,q)$ which, on restriction to $SL(m,q)$, can yield Steinberg characters of $SL(m,q)$ are the Steinberg characters of $GL(m,q)$. The Steinberg characters of $GL(m,q)$ are formed from a basic Steinberg character by multiplying by the powers of the determinant, by which we mean that if χ_0 is the basic Steinberg character $\chi_1, \ldots\ldots, \chi_{q-2}$ are defined as follows

$$\chi_i(x) = \chi_0(x)\,\varepsilon^{i\alpha}$$

where ε is a primitive (q-1)st root of unity in the complex field, ρ is a primitive (q-1)st root of unity in $GF(q)$, and $\det.x = \rho^{\alpha}$.

Thus

$$\chi_i\big|_{SL(m,q)} = \chi_0\big|_{SL(m,q)} \quad \text{for } i = 1, \ldots\ldots, q-2$$

Hence $SL(m,q)$ has at most one Steinberg character. However we know from a theorem of Curtis (4) that

PSL(m,q) has a Steinberg character, and so we can deduce that SL(m,q) has exactly one such character.

Hence we have proved

<u>Theorem 4</u> SL(m,q) has d = (m,q-1) blocks of maximal defect and one of defect zero. PSL(m,q) has one block of maximal defect and one of defect zero.

The block idempotents of SL(m,q) are as follows

$$e = \frac{1}{|SL(m,q)|}\chi(1)\sum_{\alpha}\chi(g_{\alpha}^{-1})\,K_{\alpha}$$

where χ is the Steinberg character

and $E_0 - e$, E_1, ... , E_{d-1} . e gives the block of defect zero.

The block idempotents of PSL(m,q) are similarly computed.

<u>Sp(2m,q)</u>

The centraliser of the Sylow p-subgroup of Sp(2m,q) was found in lemma 11. The only p-regular elements in this group are the scalar matrices λI. The number of such matrices is (2,q-1). Thus Sp(2m,q) has (2,q-1) blocks of maximal defect, while PSp(2m,q) has but one.

Now consider the blocks of defect zero. ~~If q is even, the centre of Sp(2m,q) is trivial, and hence Sp(2m,q) = PSp(2m,q)~~ . ~~But~~ Curtis (4) shows that ~~P~~Sp(2m,q) has exactly one character of defect zero, a~~nd hence so too does Sp(2m,q)~~ .

~~Now suppose that q is odd. Then the centre~~

~~of $Sp(2m,q)$ has order two and consists of the matrices~~ $\pm I$. Let χ be an irreducible character of defect zero i.e. $\chi(I) = q^N$ where N is the number of positive roots of C_m. Then it follows from Schur's lemma that $\chi(-I) = \pm q^N$, and whichever value it is we can obtain an irreducible character χ', with the property that $\chi'(-I)$ has the opposite sign, by simply "tensoring" χ with the representation which assigns to each matrix its determinant. Thus for every two characters of $Sp(2m,q)$ of defect zero we obtain one such character for $PSp(2m,q)$. Hence it follows from Curtis' theorem that ~~$Sp(2m,q)$ has exactly two such characters.~~

<u>$\Omega(2m,q)$, $\Omega(2m+1,q)$</u>

The reasoning for these two groups is exactly analogous to that for $Sp(2m,q)$.

Thus we have proved

<u>Theorem 5</u> $Sp(2m,q)$, $\Omega(2m,q)$, $\Omega(2m+1,q)$ each have $(2,q-1)$ blocks of maximal defect and (~~$2,q-1$~~ one) of defect zero. In each case the corresponding Chevalley group has one block of each type.

The various block idempotents are computed by the method described in theorem 3.

References

(1) R. Brauer : Zur Darstellungstheorie der Gruppen endlicher Ordnung. Math. Zeit. 63(1956) pp.406 - 444.

(2) R.W. Carter : Simple groups and simple Lie algebras. Journal of the London Math. Soc. 40(1965) pp.193 - 240.

(3) R. W. Carter : Lecture notes. University of Warwick 1967.

(4) C. W. Curtis : Irreducible representations of finite groups of Lie type. J. für Math. 219(1965) pp.180-199.

(5) C. W. Curtis : The Steinberg character of a finite group with a (B,N) pair. J. of Alg.4(1966) pp.433-441.

(6) C. W. Curtis and I. Reiner : Representation theory of finite groups and associative algebras. Interscience (1962).

(7) L. E. Dickson : Linear groups with an exposition of the Galois field theory. Leipzig (1901).

(8) J, A. Green : On the indecomposable representations of a finite group. Math. Zeit. 70(1959) pp.430 - 445.

(9) J. A. Green : Blocks of modular representations. Math. Zeit. 79(1962) pp.100 - 115.

(10) J. A. Green : Some remarks on defect groups. Math. Zeit. 107(1968) pp.133 - 150.

(11) R. Ree : On some simple groups defined by C. Chevalley. Trans. A.M.S. 84(1957) pp.392 - 400.

(12) A. Rosenberg : Blocks and centres of group algebras. Math. Zeit. 76(1961) pp.209 - 216.

(13) R. Steinberg : A geometric approach to the representations of the full linear group over a Galois field. Trans. A.M.S. 71(1951) pp.274 - 282.

(14) R. Steinberg : Representations of algebraic groups. Nagoya Math. J. 22(1963) pp.33 - 56.

(15) J. Tits : Théorème de Bruhat et sous groupes paraboliques. C.R. Acad. Sci. 254(1962) pp.2910-2912.

A Class of Irreducible Characters for Certain Classical Groups

Introduction This work arose out of an attempt to verify some of the claims made in a paper of Gelfand and Graev (4). In this they announced the following two results:- if G is a group belonging to a certain class, which consists in the main of the groups discussed in the first part of this thesis, and if P' is the derived group of the Sylow p-subgroup P

(i) $((1_{P'})^G, \chi) \geqslant 1$ for every irreducible character χ of G

(ii) $(\chi, \varphi^G) \leq 1$ for every irreducible character χ of G and every linear character φ of P whose restriction to a root subgroup of P ~~has the property that $(x_{q_i}(t)) = 1$ iff $t = 0$.~~ is nontrivial for each fundamental root

They give somewhat cursory proofs of these two theorems for the case $G = GL(m,q)$ and are content to assert their truth for the remaining cases. There have been several examples in this field where considerable difficulty has been experienced in attempts to generalise results which were true for $GL(m,q)$ to the other classical groups. In view of this and such inscrutable remarks as " in the other cases the proof goes differently", some doubt had been cast on the validity of these claims. This then is where we began and it is with (ii) and its consequences that we shall be concerned. The first part of our investigation is negative, insofar as it serves only to justify the assertions of Gelfand and Graev. However as a result of the techniques that we develop to do this, we are able to find a class of irreducible characters for the groups concerned. The major part of what follows is concerned with the construction and properties of these characters.

<u>Notation</u> Let G be a group belonging to one of the ten classes described in the first part of this thesis i.e. $GL(m,q)$, $PGL(m,q)$, $Sp(2m,q)$ etc. Let P be its Sylow p-subgroup, and let χ be a linear character of P. Let e_χ be the uniquely-determined, central idempotent of the complex group algebra CP which affords the character χ.

$$\text{i.e.} \qquad e_\chi = \frac{1}{|P|} \sum_{u \in P} \chi(u^{-1})\, u$$

We shall say that χ is of <u>general aspect</u> if ~~$\chi(x_{q_i}(t)) = 1$ implies that $t = 0$ for every fundamental root q_i of the Lie algebra which corresponds to G.~~ χ is non trivial on each Fundamental root subgroup

§1

We are interested in the module $CGe_\chi \cong (CPe_\chi)^G$. In particular we shall consider its endomorphism ring. Since $G = PNP$ and CPe_χ is one-dimensional, a basis for this endomorphism ring is provided by the set

$\{e_\chi n e_\chi$; n belongs to $N\}$.

By no means all of the $e_\chi n e_\chi$ are non-zero, and it is the purpose of this section to find the conditions under which this happens. At this point I must again acknowledge my indebtedness to Professor Green. The present proof of lemma 1 is his and replaces an earlier, somewhat heuristic, proof of my own.

We first observe that for any u belonging to P

$$ue_\chi = e_\chi u = \chi(u)\, e_\chi \qquad (1.1)$$

and hence

$$e_\chi n e_\chi = \chi(u_1)^{-1}\chi(u_2)^{-1}\, u_1 e_\chi n e_\chi u_2 \qquad (1.2)$$

Let the coefficient of n in $e_\chi n e_\chi$ be α. Once α has been specified, equation (1.1) enables us to determine every coefficient in $e_\chi n e_\chi$.

Now suppose that $u_1 n u_2 = n$. Then if we compare the coefficients of n in (1.2) we find that

$$\alpha = \alpha\, \chi(u_1)^{-1}\chi(u_2)^{-1} \qquad (1.3)$$

(1.1) implies that $e_\chi n e_\chi \neq 0$ iff $\alpha \neq 0$.

Therefore (1.3) enables us to deduce that $e_\chi n e_\chi \neq 0$ implies that for every u belonging to $P \cap P^n$

$\chi(u) = \chi(nun^{-1})$.

Conversely suppose that for every u belonging to $P \cap P^n$ $\chi(u) = \chi(nun^{-1})$. Then the coefficient of n in $e_\chi n e_\chi$ is

$$\frac{1}{|P|^2} \sum_{\substack{u_1, u_2 \in P \\ \text{s.t. } u_1 n u_2 = n}} \chi(u_1)^{-1} \chi(u_2)^{-1}$$

But $u_1 n u_2 = n$ implies that $u_2^{-1} = n^{-1} u_1 n$.

Therefore
$$\chi(u_1)^{-1} \chi(u_2)^{-1} = \chi(u_1^{-1}) \chi(n^{-1} u_1 n)$$
$$= 1$$

Hence the coefficient of n in $e_\chi n e_\chi$ is $\dfrac{|P \cap P^n|}{|P|^2}$

Thus we have proved the following lemma

<u>Lemma 1</u> (i) $e_\chi n e_\chi \neq 0$ iff for every u belonging to $P \cap P^n$, $\chi(u) = \chi(nun^{-1})$.

(ii) If $e_\chi n e_\chi \neq 0$, the coefficients of the elements which appear in the sum $e_\chi n e_\chi$ are determined by the conditions

(a) coefficient of n $= \dfrac{|P \cap P^n|}{|P|^2}$

(b) $u e_\chi = e_\chi u = \chi(u) e_\chi$ for every u belonging to P.

As with the earlier work this may be interpreted in terms of the root structure of the corresponding Lie algebra.

We shall now suppose that χ is of general aspect. Since we are interested in χ^G rather than χ itself, we may assume without loss of generality that $\chi(x_{q_i}(t)) = \chi(x_{q_1}(t))$ for every fundamental root q_i.

This assumption, together with the realisation that χ is trivial on P', implies that part 1 of lemma one implies the following condition on w

$$\begin{array}{ll} & w(q_i) > 0 \text{ implies that } w(q_i) = q_j \text{ for every} \\ & \qquad\qquad\qquad\qquad \text{fundamental root } q_i \\ (1.4) & \\ & w(r) = \pm s \text{ or } -q_j \text{ where } r, s \text{ are positive} \\ & \qquad\qquad\qquad\qquad \text{but not fundamental roots} \end{array}$$

§2

In this section we look at the various root systems and find those w which satisfy condition (1.4). The root systems were described in §2 of the first part of this thesis, and so we shall not repeat them here.

A_m

Suppose that w satisfies condition (1.4)

$w(a_m) = a_s$ for some s

If $s \neq 0$, there exists an h such that $w(a_h) = a_{s-1}$.

Therefore $w(a_h - a_m) = a_{s-1} - a_s$

$a_h - a_m$ is positive and $a_{s-1} - a_s$ is fundamental. Hence (1.4) implies that $h = m-1$.

Similarly we have that either $s-1 = 0$ or $w(a_{m-2}) = a_{s-2}$.

Proceeding in this way we obtain

Lemma 2 If L is a Lie algebra of type A_m, the w which satisfy (1.4) are precisely those which, when written as permutations, have the form

$$\left(\begin{array}{cccc|cccc|cccc} 0 & 1 & 2 & \cdots\cdots & m-t_1 & \cdots & m-t-2 & m-t-1 & m-t & \cdots & m-1 & m \\ & & & & t+1 & \cdots & t_1-1 & t_1 & 0 & & t-1 & t \end{array}\right)$$

where t, $t_1, \ldots$ are some integers such that $m \geqslant \ldots\ldots > t_1 > t \geqslant 0$.

B_m

There exists an a_i such that $w(a_i) = \pm a_1$. Hence we may distinguish two cases

Case 1 $w(a_i) = a_1$

Suppose that there exists an h with the property that $w(a_n) = -a_h$ for some integer n. Then there exists a t such that $w(a_t) = a_{h-1}$ (We are assuming that h is the smallest integer with the given property.)

Therefore $w(a_n + a_t) = a_{h-1} - a_h$

which contradicts (1.4)

Therefore $w(a_j) = a_{w(j)}$ for every j

But now a consideration of the fundamental roots a_m, $a_{m-1} - a_m$, ... etc. enables us to deduce that $w(a_j) = a_j$ for every j.

<u>Case 2</u> $\quad w(a_i) = -a_1$

Suppose first that $i > 1$. Consider in turn the fundamental roots $a_{i-1} - a_i$, ..., $a_1 - a_2$.

$$w(a_{i-1} - a_i) = a_1 + w(a_{i-1})$$

Therefore (1.4) implies that $w(a_{i-1}) = -a_2$.

Proceeding in this way we obtain $w(a_{i-r}) = -a_{r+1}$ for $r = 0, 1, \ldots\ldots, i-1$.

Now consider $a_{i+1}, \ldots\ldots, a_m$. As before there exists a_j with the property that $w(a_j) = \pm a_{i+1}$.

The arguments above tell us that

$$\begin{cases} w(a_j) = a_{i+1} \text{ implies that } w(a_h) = a_h \text{ for every } h > i \\ w(a_j) = -a_{i+1} \text{ implies that } w(a_{j-1}) = -a_{i+2} \text{ etc.} \end{cases}$$

We observe further that if there exists an a_α with the property that $w(a_\alpha) = -a_\beta$ for some β, $w(a_\gamma) < 0$ for every $\gamma < \alpha$. To prove this consider $a_{\alpha-1} - a_\alpha$.

$w(a_{\alpha-1} - a_\alpha) = w(a_{\alpha-1}) + a_\beta$ and hence $w(a_{\alpha-1}) < 0$ etc.

If $w(a_1) = -a_1$, let h be the least integer such that $w(a_h) \neq -a_h$ and repeat the above argument using $a_h, \ldots, a_m$ instead of $a_1, \ldots, a_m$.

The above argument works for C_m and with one or two minor modifications for D_m. That is in each case we establish the following facts

(i) If there exists an a_j such that $w(a_j) = -a_i$ for some i, $w(a_h) < 0$ for every $h \leq j$.

(ii) $w(a_i) = a_1$ for some i implies that $w(a_j) = a_j$ for every j.

(iii) $w(a_i) = -a_1$ for some $i > 1$ implies that $w(a_{i-r}) = -a_{r+1}$ for $r = 0, 1, \ldots, i-1$.

Hence we have the following lemma

<u>Lemma 3</u> If L is a Lie algebra of type B_m, C_m, or D_m, the w which satisfy (1.4) are precisely those of the form

$$\left(\begin{array}{c|c|c|c|c} a_1 \cdots\cdots a_{h-1} & a_h \cdots\cdots a_i & a_{i+1} \cdots\cdots a_j & \cdots\cdots & a_t \cdots\cdots a_m \\ -a_1 \cdots\cdots -a_{h-1} & -a_i \cdots\cdots -a_h & -a_j \cdots\cdots -a_{i+1} & \cdots\cdots & a_t \cdots\cdots a_m \end{array}\right)$$

where i, j, h, t are some integers $1 \leq h \leq i < j \ldots\ldots t \leq m+1$

<u>§3</u>

It was convenient first to consider condition (1.4), but of course, lemma 1 imposes more stringent conditions on n than that under the homomorphism $N \to W$ it should correspond to a w of the type described in lemmas 2 and 3.

In this section we shall determine the precise conditions on n in order that $e_\chi n e_\chi \neq 0$.

<u>$GL(m,q)$</u>

We consider $GL(m,q)$ rather than $SL(m,q)$, since, if the theorem is true for the former group, it is certainly true for the latter.

The n corresponding to the w of lemma 2 have the form

$$n = \left[\begin{array}{cc|c}
 & 0 & \begin{matrix} y_1 & & 0 \\ & y_2 & \\ 0 & & \ddots \\ & & & y_r \end{matrix} \\ \hline
 & \begin{matrix} z_1 & & 0 \\ & z_2 & \\ 0 & & \ddots \\ & & & z_s \end{matrix} & \\
 \iddots & & 0
\end{array}\right]$$

The element $x_{q_i}(t)$ corresponds to the matrix $I + tE_{i,i+1}$.

Lemma 1 part (i) says that

$$(3.1) \qquad n\, x_{q_i}(t)\, n^{-1} = x_{w(q_i)}(t) \quad \text{for every } q_i \text{ such that } w(q_i) = q_j \text{ .}$$

An easy calculation now shows that this is equivalent to

$$y_i = y_1 \text{ for all } i$$
$$z_j = z_1 \text{ for all } j$$
$$\text{etc.}$$

<u>Sp(2m,q)</u>

The n corresponding to the w of lemma 3 have the form

$$n = \left[\begin{array}{cccc|cccc}
 & & & & \begin{matrix} y_1 & & 0 \\ & \cdot & \\ 0 & & y_\alpha \end{matrix} & & & \\
 & 0 & & & & \begin{matrix} 0 & & z_1 \\ & \cdot & \\ z_\beta & & 0 \end{matrix} & & 0 \\
 & & & & & 0 & \cdot & \\
 & & & \begin{matrix} f_1 & & 0 \\ & \cdot & \\ 0 & & f_\nu \end{matrix} & & & & 0 \\
\hline
\begin{matrix} -y_1^{-1} & & 0 \\ & \cdot & \\ 0 & & -y_\alpha^{-1} \end{matrix} & & 0 & & & & & \\
 & \begin{matrix} 0 & & -z_1^{-1} \\ & \cdot & \\ -z_\beta^{-1} & & 0 \end{matrix} & & & & 0 & & \\
 & & \cdot & & & & & \\
0 & & & 0 & & & & \begin{matrix} f_1^{-1} & & 0 \\ & \cdot & \\ 0 & & f_\nu^{-1} \end{matrix}
\end{array}\right]$$

If we number the rows and columns from 1 to m and then -1 to -m , the matrices corresponding to the fundamental roots are

$$x_{q_i}(t) \longleftrightarrow \quad I + t(E_{i,i+1} + E_{-(i+1),-i}) \text{ for } i = 1,\ldots,m-1$$
$$x_{q_m}(t) \longleftrightarrow \quad I + t\,E_{m,-m}$$

An easy calculation shows that (3.1) is equivalent to

$$y_i = (-1)^{i+1}\, y_1 \quad \text{for every } i$$
$$z_j = (-1)^{j+1}\, z_1 \quad \text{for every } j$$
$$\text{etc.}$$
$$f_h = f_1 = \pm 1 \quad \text{for every } h$$

<u>$\Omega(2m,q)$</u>

This is very similar to $Sp(2m,q)$, save that in the expression for n the minus signs in the bottom left hand quadrant are removed, and the matrices corresponding to the fundamental roots are

$$x_{q_i}(t) \longleftrightarrow I + t(E_{i,i+1} - E_{-(i+1),-i}) \text{ for } i = 1,..,m-1$$
$$x_{q_m}(t) \longleftrightarrow I + t(E_{m-1,-m} - E_{m,-(m-1)})$$

The final result is identical i.e. $y_i = (-1)^{i+1}\, y_1$, etc.

<u>$\Omega(2m+1,q)$</u>

If we number the rows and columns 0 to m and then -1 to -m , the matrices corresponding to the fundamental roots are as follows:-

$$x_{q_i}(t) \longleftrightarrow I + t(E_{i,i+1} - E_{-(i+1),-i}) \text{ for } i = 1,..,m-1$$
$$x_{q_m}(t) \longleftrightarrow I + t(E_{0,-m} - 2E_{m,0}) - t^2 E_{m,-m}$$

The matrices n corresponding to the w of lemma 3 have only one non-zero entry in the 0th row and column, this being in the (0,0)th place. The remaining minor is of the form described for $\Omega(2m,q)$.

As with the previous two groups (3.1) is equivalent to

$$y_i = (-1)^{i+1} y_1 \quad \text{for every } i$$

etc.

$$f_h = f_1 = \pm 1 \quad \text{for every } h$$

§4

In this section we show how to define anti-automorphisms θ for each group. These anti-automorphisms have the properties

(i) $\theta : x_{q_i}(t) \rightarrow x_{q_{j(i)}}(t)$ for every q_i

(ii) $\theta : n \rightarrow n$ for every n such that $e_\chi n e_\chi \neq 0$.

The existence of these maps is, of course, the crux of the whole argument, and Gelfand and Graev's proof for the case of GL(m,q) consists, in fact, of little more than a description of θ and its properties.

As in section 3 we shall be content to consider GL(m,q), Sp(2m,q), Ω(2m,q), and Ω(2m+1,q), since, if the theorem is true for these four classes, it is automatically true for the other six.

GL(m,q)

Let θ be the composite map given by

(i) transposition

(ii) conjugation by $$\begin{bmatrix} & & 1 \\ 0 & \cdot & \\ & \cdot & 0 \\ 1 & & \end{bmatrix}$$

$Sp(2m,q)$

Let θ be the composite map given by

(i) transposition

(ii) conjugation by $\left[\begin{array}{c|c} & I_m \\ \hline -I_m & \end{array}\right]$

(iii) conjugation by (λ_{ii}) where $\lambda_{ii} = (-1)^{i+1}$ if $i > 0$ and $\lambda_{ii} = (-1)^{i}$ if $i < 0$.

$\Omega(2m,q)$

Let θ be the composite map given by

(i) transposition

(ii) conjugation by $\left[\begin{array}{c|c} & I_m \\ \hline I_m & \end{array}\right]$

(iii) conjugation by (λ_{ii}) where $\lambda_{ii} = (-1)^{i+1}$ for all i .

$\Omega(2m+1,q)$

Let θ be the composite map given by

(i) transposition

(ii) conjugation by $\left[\begin{array}{c|c|c} 1 & & \\ \hline & & I_m \\ \hline & I_m & \end{array}\right]$

(iii) conjugation by (λ_{ii}) where $\lambda_{00} = 1/_2$,

$\lambda_{ii} = (-1)^{m+1-i}$ for all $i \neq 0$.

Verifying that in each case θ has the required property is simply a matter of checking e.g. for $\Omega(2m,q)$ we have

$$\theta : \quad I + t(E_{i,i+1} - E_{-(i+1),-i}) \xrightarrow{\text{transp.}} I + t(E_{i+1,i} - E_{-i,-(i+1)})$$

$$\xrightarrow{\text{conj.}} I + t(E_{-(i+1),-i} - E_{i,i+1}) \xrightarrow{\text{conj.}} I + t(E_{i,i+1} - E_{-(i+1),-i})$$

$$\theta : \quad I + t(E_{m-1,-m} - E_{m,-(m-1)}) \xrightarrow{\text{transp.}} I + t(E_{-m,m-1} - E_{-(m-1),m})$$

$$\xrightarrow{\text{conj.}} I + t(E_{m,-(m-1)} - E_{m-1,-m}) \xrightarrow{\text{conj.}} I + t(E_{m-1,-m} - E_{m,-(m-1)})$$

Finally θ maps the element n of section 3 on to

$$\left[\begin{array}{cccc|cccc}
 & & & & \begin{matrix} y_1 & & \\ & \cdot & \\ & & y_\alpha \end{matrix} & & & \\
 & & & & & \begin{matrix} & & dz_\beta \\ & \cdot & \\ dz_1 & & \end{matrix} & & \\
 & & & & & & \cdot\ \cdot & \\
 & & & \begin{matrix} f_1^{-1} & & \\ & \cdot & \\ & & f_\nu^{-1} \end{matrix} & & & & 0 \\
\hline
\begin{matrix} y_1^{-1} & & \\ & \cdot & \\ & & y_\alpha^{-1} \end{matrix} & & & & & & & \\
 & \begin{matrix} & & dz_\beta^{-1} \\ & \cdot & \\ dz_1^{-1} & & \end{matrix} & & & & & & \\
 & & \cdot\ \cdot & & & & & \\
 & & & 0 & & & & \begin{matrix} f_1 & & \\ & \cdot & \\ & & f_\nu \end{matrix}
\end{array}\right]$$

where $d = (-1)^{\beta+1}$

The conditions $\quad z_i = (-1)^{i+1} z_1$ for all i

etc.

$$f_j = f_1 \text{ for all } j$$

which were established in section 3, show that this element is the same as the original.

The verifications for $Sp(2m,q)$ and $\Omega(2m+1,q)$ are similar.

§5

We are now in a position to verify Gelfand and Graev's second assertion for the ten classes of groups under consideration.

Theorem 1 Let χ be a character of general aspect and G a group of the type listed earlier. Then if ψ is any irreducible character of G $(\chi^G, \psi) = 0$ or 1.

Proof This is equivalent to shwing that the endomorphism ring, $e_\chi CGe_\chi$, of CGe_χ is commutative.

Since a basis for $e_\chi CGe_\chi$ is provided by the non-zero elements in the set $\{ e_\chi n\, e_\chi ;$ n belongs to N $\}$, it suffices to show that if $e_\chi n_1 e_\chi$ and $e_\chi n_2 e_\chi$ are non-zero.

$$e_\chi n_1 e_\chi \cdot e_\chi n_2 e_\chi = e_\chi n_2 e_\chi \cdot e_\chi n_1 e_\chi \ .$$

$$e_\chi n_1 e_\chi \cdot e_\chi n_2 e_\chi = \sum_{n\in N} \alpha_n e_\chi n\, e_\chi \text{ where } \alpha_n \text{ is some element of C .}$$

Apply θ to this last equation.

$$e_{\chi}^{\theta} n_2^{\theta} e_{\chi}^{\theta} n_1^{\theta} e_{\chi}^{\theta} = \sum_{n \in N} \alpha_n e_{\chi}^{\theta} n^{\theta} e_{\chi}^{\theta} .$$

But property (i) of θ implies that $e_{\chi}^{\theta} = e_{\chi}$, and so combining this with θ's other property we find that

$$e_{\chi} n_2 e_{\chi} n_1 e_{\chi} = \sum_{n \in N} \alpha_n e_{\chi} n e_{\chi} \quad \text{as required.}$$

§6

In this section we construct a family of irreducible characters for the groups under consideration. We do this by supposing that χ, ψ are linear characters of P and then finding common constituents of χ^G and ψ^G. This entails a study of the module $\mathrm{Hom.}_{CG}(CGe_{\chi}, CGe_{\psi})$. $\mathrm{Hom.}_{CG}(CGe_{\chi}, CGe_{\psi}) \cong e_{\chi}CGe_{\psi}$, and this latter is clearly spanned by the set $\{e_{\chi} n e_{\psi};\ n \text{ belongs to } N\}$. Hence we are again concerned to know those n for which $e_{\chi} n e_{\psi} \neq 0$.

Subject to the obvious modifications, the argument of section one carries through to produce the following result.

Lemma 4 (i) $e_{\chi} n e_{\psi} \neq 0$ iff for every u belonging to $P \cap P^n$, $\chi(u) = \psi(nun^{-1})$.

(ii) If $e_{\chi} n e_{\psi} \neq 0$, the coefficients of the elements which appear in the sum $e_{\chi} n e_{\psi}$ are determined by the conditions

(a) coefficient of n $= \dfrac{|P \cap P^n|}{|P|^2}$

(b) $u\, e_\chi = e_\chi u = \chi(u)\, e_\chi$ for every $u \in P$.

(c) $u\, e_\psi = e_\psi u = \psi(u)\, e_\psi$ for every $u \in P$.

Thus we now have a good description of $\mathrm{Hom.}_{CG}(CGe_\chi, CGe_\psi)$ and the next thing we need to be able to do is to determine those images of CGe_χ which are irreducible. The general element of $\mathrm{Hom.}_{CG}(CGe_\chi, CGe_\psi)$ has the form $e_\chi \sum_{n_i \in N} \lambda_i n_i e_\psi$ where λ_i belongs to C.

Lemma 5 $CGe_\chi(\sum_{n_i \in N} \lambda_i n_i)\, e_\psi$ is an irreducible CG-submodule of CGe_ψ if and only if the module $e_\psi CGe_\chi(\sum_{n_i \in N} \lambda_i n_i)\, e_\psi$ has dimension one over C.

Proof Let $e_\psi = e_1 + \ldots\ldots + e_r$ be a decomposition of e_ψ into primitive idempotents of CG, and let $\nu = \sum_{n_i \in N} \lambda_i n_i$.

Suppose that $CGe_\chi \nu e_\psi$ is an irreducible submodule of CGe_ψ.

Then $CGe_\chi \nu e_\psi \cong CGe_i$ for some i.

Therefore $e_\psi CGe_\chi \nu e_\psi \cong (e_1 + \ldots\ldots + e_r)\, CGe_i$.

But since CGe_ψ contains no irreducible character more than once, — (we are supposing that ψ is of general aspect) -, $e_j CGe_i = 0$ unless $j = i$ (1).

Therefore $e_\psi CGe_\chi \nu e_\psi \cong e_i CGe_i \cong C$.

Conversely if $CGe_\chi \nu e_\psi$ is not irreducible

$$CGe_{\chi}\nu e_{\psi} \cong CG\,(e_{\alpha} + \ldots\ldots + e_{\kappa})$$ for some subset of the e_i whose cardinal is >1.

Therefore $\quad e_{\psi}CGe_{\chi}\nu e_{\psi} \cong e_{\alpha}CGe_{\alpha} \oplus \ldots\ldots \oplus e_{\kappa}CG\,e_{\kappa}$.

Therefore $\quad$ dim. > 1 .

Thus for a given ν we have reduced the problem to a consideration of the size of the set $\{ e_{\psi}n\, e_{\chi}\nu e_{\psi};\ \ n \in N \}$.

So far we have managed to avoid imposing any restrictions on χ . However it is now more profitable if we take a particular χ , especially if we have the good sense to choose χ to be the trivial character on P. Although this is of course the easiest case at our disposal, it does yield a fairly large class of characters, and goes a considerable way to achieving a complete decomposition of the character $(1_P)^G$. It was a consideration of this latter character which enabled Curtis to prove the existence of a Steinberg character for any finite group having a (B,N)-pair (3) .

Thus we shall henceforth suppose that $\chi = 1_P$, i.e. that $\quad e_{\chi} = \frac{1}{|P|}\sum_{u\in P} u$.

The first step is to determine those n for which $e_{\chi}n\,e_{\psi} \neq 0$.

Lemma 4 implies that $e_{\chi}n\,e_{\psi} \neq 0$ iff $\psi(nun^{-1}) = 1$ for every u belonging to $P \cap P^{n}$.

$$P \cap P^n = Gp.\left\{x_r(t)\ ;\ r \in \Pi^+,\ w(r) \in \Pi^+,\ t \in GF(q)\right\}.$$

But, since ψ is of general aspect, $\psi(x_r(t)) = 1$ for every t belonging to GF(q) implies that r is not a fundamental root, and so for no fundamental root r can we have w(r) belonging to Π^+.

Therefore $w = w_o$.

Choose an element n_o belonging to N which corresponds to w_o under the homomorphism $N \to W$.

Then the set of n which satisfy the condition $e_\chi n e_\psi \neq 0$ is precisely the set $\{hn_o\ ;\ h \text{ belongs to } H\}$.

We get the same result for the set of n which satisfy the condition $e_\psi n e_\chi \neq 0$.

Hence in order to find the common, irreducible constituents of χ^G and ψ^G, we have to find elements η of CH such that the C-module spanned by the set

$\{e_\psi h n_o e_\chi n_o \eta e_\psi\ ;\ h \text{ belongs to } H\}$ has dimension one.

$$\begin{aligned} h n_o e_\chi n_o \eta &= n_o n_o^{-1} h n_o e_\chi n_o \eta \\ &= n_o e_\chi n_o n_o^{-2} h n_o^2 \eta \\ &= n_o e_\chi n_o h \eta \quad \text{since } n_o^2 \text{ belongs to H.} \end{aligned}$$

Therefore, since H is Abelian, a necessary and sufficient condition on η is that $\eta = \varepsilon_\varphi$, the idempotent in CH which corresponds to an irreducible character φ of H.

Thus we have proved, or at least partly proved

Lemma 6 If χ is the trivial character on P, ψ any character of general aspect, φ any irreducible character of H and e_χ, e_ψ, ε_φ the corresponding idempotents in CP, CH respectively, the CG-module $CG\, e_\chi n_o \varepsilon_\varphi e_\psi$ is irreducible.

Proof The above argument shows that the module is irreducible provided it is non-zero. To see that it is non-zero consider the element $e_\chi n_o \varepsilon_\varphi e_\psi$. This belongs to our module, and is a sum of terms of the form $\lambda u_1 n_o h u_2$ where λ belongs to C. Each product $u_1 n_o h u_2$ occurs but once, and $u_1 n_o h u_2 = \bar{u}_1 n_o \bar{h} \bar{u}_2$ implies that $u_1 = \bar{u}_1$, $h = \bar{h}$, $u_2 = \bar{u}_2$. Hence our module is non-zero.

Lemma 6, as it stands, is not very valuable. If we are looking for irreducible characters of G, to be told that $CG\, e_\chi n_o \varepsilon_\varphi e_\psi$ provides one is almost a case of the cure being worse than the disease. To render it useful we need to find the idempotent of this module. Once we have this we can then write down the character by means of the following lemma:-

Lemma 7 (5) Suppose that $R^1, \ldots, R^\mu$ are the distinct irreducible representations of G, and suppose that $e = \sum_{g \in G} \lambda_g g$ is a primitive idempotent of CG with the property that $CGe \cong R^\alpha$.

Then if K_e is a conjugacy class of G

$$\chi_e^{\alpha} = \frac{|G|}{h_e} \sum_{g \in \overline{K}_e} \lambda_g$$

where $\overline{K}_e = \{ g \; ; \; g^{-1} \text{ belongs to } K_e \}$ and h_e is the cardinal of K_e.

<u>Proof</u> $\qquad R^{\beta}(e) = \sum_{g \in G} \lambda_g R^{\beta}(g)$

If $\beta \neq \alpha$ $\; R^{\beta}(e) = 0$, since $e \, CG \, e_{\beta} = 0$.

$R^{\alpha}(e)$ is a matrix whose only diagonal term is a 1 in the top left hand corner. To see this let $e, \gamma_1 e, \ldots\ldots \gamma_t e$ be a basis for CGe. Then if $e \gamma_i e = \lambda \gamma_i e$, we must have a contradiction, since $e \gamma_i e$ belongs to $\text{End.}_{CG}(CGe) \cong C$ and is hence equal to λe for some λ belonging to C.

Therefore $\sum_{g \in G} \lambda_g R^{\beta}(g) \begin{cases} = 0 & \text{if } \beta \neq \alpha \\ = \begin{bmatrix} 1 & & \\ & 0 & * \\ & * & \ddots & \\ & & & 0 \end{bmatrix} & \text{if } \beta = \alpha \end{cases}$

Taking traces we get

$$\delta_{\beta\alpha} = \sum_e \Lambda_e \chi_e^{\beta} \quad \text{for } \beta = 1, \ldots\ldots, \mu$$

where $\Lambda_e = \sum_{g \in K_e} \lambda_g$

Therefore $\displaystyle \sum_{\beta} \delta_{\beta\alpha} \chi_{\sigma}^{\beta'} = \sum_{\beta} \sum_e \Lambda_e \chi_e^{\beta} \chi_{\sigma}^{\beta'}$ $\quad$ ($\chi^{\beta'}$ = conjugate of χ^{β})

$$= \sum_e \Lambda_e \frac{|G|}{h_e} \delta_{e\sigma}$$

Therefore $\chi_e^{\alpha'} = \frac{|G|}{h_e} \Lambda_e$

Therefore $\chi_e^{\alpha} = \frac{|G|}{h_e} \sum_{g \in \bar{K}_e} \lambda_g$

We now find the primitive idempotent of CG which yields the module of lemma 6.

If e is a primitive idempotent of CG, the irreducible module CGe has CG-endomorph ism ring eCGe isomorphic to C. That is to say any element of the CG-endomorphism ring has the form λ e for some λ belonging to C, and e may be identified as being the only non-zero idempotent element of this form.

The CG-endomorphism ring of CG $e_\chi n_0 \varepsilon_\varphi e_\psi$ has C-basis $e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$.

Therefore the idempotent of CG $e_\chi n_0 \varepsilon_\varphi e_\psi$ must be of the form $\alpha\, e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$ for some α belonging to C.

To find α we compare the coefficients of 1 in $\alpha\, e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$ and its square.

$$e_\chi n_0 \varepsilon_\varphi e_\psi = \sum_{\substack{u_1, u_2 \in P \\ h \in H}} \frac{1}{|P|^2} \cdot \frac{1}{|H|} \varphi(h^{-1}) \psi(u_2^{-1}) u_1 n_0 h u_2$$

As we remarked earlier, the elements $u_1 n_0 h u_2$ are all distinct, and hence so too are the elements $n_0 u_1 n_0 h u_2$.

Thus in finding thevcoefficient of 1 in $\alpha\, e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$

we shall be interested in precisely those elements of the form $n_0 u_1 n_0 h u_2$ which lie in P.

$n_0 u_1 n_0 h u_2$ belongs to P only if $n_0 u_1 n_0$ belongs to $B = HP$ i.e. only if $n_0 u_1 n_0^{-1}$ belongs to B. But $n_0 u_1 n_0^{-1}$ belongs to B if and only if $u_1 = 1$, and hence we deduce that $n_0 u_1 n_0 h u_2$ belongs to P if and only if $u_1 = 1$ and $h = n_0^{-2}$.

Therefore the coefficient of 1 in $\alpha e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$

$$= \alpha \frac{1}{|P|^3} \cdot \frac{1}{|H|} \sum_{u \in P} \psi(u^{-1}) \varphi(n_0^2) \psi(u)$$

$$= \alpha \frac{1}{|P|^2} \cdot \frac{1}{|H|} \varphi(n_0^2)$$

More generally, if we let $\beta_g = \sum_{\substack{u_1, h, u_2, u_3 \text{ s.t.} \\ u_1 n_0 u_2 n_0 h u_3 = g}} \psi(u_1^{-1}) \varphi(h^{-1}) \psi(u_3^{-1})$

the coefficient of g in $\alpha e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi$ is

$$\alpha \cdot \frac{1}{|P|^3} \cdot \frac{1}{|H|} \beta_g$$

Hence the coefficient of 1 in $(\alpha e_\psi n_0 e_\chi n_0 \varepsilon_\varphi e_\psi)^2$

$$= \frac{\alpha^2}{|P|^6} \cdot \frac{1}{|H|^2} \sum_{g \in G} \beta_g \beta_{g^{-1}}$$

whence we find that

$$\alpha = \frac{|P|^4 |H| \varphi(n_0^2)}{\sum_{g \in G} \beta_g \beta_{g^{-1}}}$$

Thus by means of lemma 7 we can now write down the value of the character at the various classes of G.

In particular its degree is $\dfrac{|G|\ |P|^2\phi(n_0^4)}{\sum_{g\in G}\beta_g\beta_{g^{-1}}}$

The drawback to this is, of course, the innocent-looking β_g. For a particular group this will not be too troublesome, but an attempt to write down a general formula will inevitably get bogged down. However, a few general observations are in order.

(i) It is only necessary to find β_n for each n belonging to N, since $\beta_{u_1 n u_2} = \psi(u_1^{-1}u_2^{-1})\beta_n$.

(ii) If there exists a u belonging to $P \cap P^n$ such that $\psi(u) \neq \psi(mun^{-1})$, $\beta_n = 0$.

<u>Proof</u> If such a u does exist lemma 1 implies that

$e_\psi n\, e_\psi = 0$.

Therefore for every v belonging to the double coset PnP

$$\sum_{\substack{v_1, v_2 \in P \\ s.t.\ v_1 n v_2 = v}} \psi(v_1^{-1})\psi(v_2^{-1}) = 0 .$$

This is a sum of roots of unity. Therefore if we take the sum which consists of the inverse of each of the above roots of unity, this will also be zero.

i.e.

$$\sum_{\substack{v_1, v_2 \in P \\ s.t.\ v_1 n v_2 = v}} \psi(v_1)\psi(v_2) = 0$$

$$\beta_n = \sum_{\substack{u_1, u_2, h, u_3 \\ s.t.\ u_1 n_0 u_2 n_0 h u_3}} \psi(u_1^{-1})\phi(h^{-1})\psi(u_3^{-1})$$

If we denote $n_0 u_2 n_0 h$ by v, v belongs to the double coset PnP.

But $\beta_n = \sum_{\substack{u_2, h \text{ s.t.} \\ \exists u_1, u_3 \text{ for} \\ \text{which } u_1 v u_3 = n}} \phi(h^{-1}) \left(\sum_{\substack{u_1, u_3 \text{ s.t.} \\ u_1^{-1} n u_3^{-1} = v}} \psi(u_1^{-1}) \psi(u_3^{-1}) \right)$

and so by our above remarks $\beta_n = 0$.

Thus the analysis of sections 2 and 3 enables us to eliminate most elements of N and hence of G.

(iii) If we denote the group $n_0 P n_0^{-1}$ by Q, the problem of finding the coefficients β_n may be reduced to that of writing each element of QH in the form $u_1 n u_2$ for some $n \in N$, $u_1, u_2 \in P$.

<u>Proof</u> Since G = PQHP there is at least one element of QH in each double coset PnP.

Choose a particular double coset, and suppose that vh is one of the elements of QH which lies in it.

There exist u_1, u_2 such that $vh = u_1 n u_2$.

Hence $n = u_1^{-1} vh u_2^{-1}$, and this latter is an expression of the form which appears in the above sum for β_n.

u_1, u_2 are not uniquely determined, the degree of choice being governed by $P \cap P^n$, whence the contribution to β_n of this particular vh is

$$\psi(u_1)\psi(u_2)\phi(n_0^2 h^{-1}) \sum_{u \in P \cap P^n} \psi(u)\psi(n u^{-1} n^{-1})$$

Thus in view of observation (ii) if $\beta_n \neq 0$

$$\beta_n = \sum_{\substack{\text{all } vh \\ \text{belonging to } PnP}} |P \cap P^n| \psi(u_1)\psi(u_2)\phi(n_0^2 h^{-1})$$

where v is an element of Q , and u_1, u_2 elements of P such that $vh = u_1 n u_2$.

We now turn our attention to a consideration of the number of characters obtainable in this way i.e. to the number of irreducible characters of G which occur as constituents both of $(1_P)^G$ and ψ^G for some character ψ of general aspect.

<u>Lemma 8</u> If φ_1, φ_2 are two irreducible characters of H, and if χ, ψ, e_χ, e_ψ, ε_{φ_1}, ε_{φ_2} are as defined in lemma 6, the irreducible left CG-modules $CG\, e_\chi n_o \varepsilon_{\varphi_1} e_\psi$ and $CG\, e_\chi n_o \varepsilon_{\varphi_2} e_\psi$ are isomorphic if and only if φ_1 and φ_2 are conjugate in G.

<u>Proof</u> Suppose that $CG\, e_\chi n_o \varepsilon_{\varphi_1} e_\psi \cong CG\, e_\chi n_o \varepsilon_{\varphi_2} e_\psi$

Then there exists a CG-isomorphism η from $CG\, e_\chi n_o \varepsilon_{\varphi_1} e_\psi$ to $CG\, e_\chi n_o \varepsilon_{\varphi_2} e_\psi$.

Therefore there exists γ belonging to CG such that

$$e_\chi n_o \varepsilon_{\varphi_1} e_\psi \overset{\eta}{\longmapsto} \gamma e_\chi n_o \varepsilon_{\varphi_2} e_\psi .$$

γ is not, of course, unique. What we shall do is to show that it can be chosen to be of the form $e_\chi n$ for some n belonging to N, i.e. that there exists an n belonging to N such that

$$e_\chi n_o \varepsilon_{\varphi_1} e_\psi \overset{\eta}{\longmapsto} e_\chi n\, e_\chi n_o \varepsilon_{\varphi_2} e_\psi .$$

It will then follow that ϕ_1, ϕ_2 are conjugate.

For every u belonging to P, $u\, e_\chi n_0 \varepsilon_{\phi_1} e_\psi = e_\chi n_0 \varepsilon_{\phi_1} e_\psi$.

Therefore for every u belonging to P,

$$u \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi = \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi$$

Let $1 = \sum_i e_i$ be a decomposition into primitive idempotents of CP, and let λ_i be the character afforded by the module CPe_i.

Then $\gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi = \sum_i e_i \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi$

Therefore $u \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi = \sum_i \lambda_i(u)\, e_i \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi$, for every u belonging to P.

Hence for every u belonging to P

$$\sum_i e_i \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi = \sum_i \lambda_i(u)\, e_i \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi.$$

Thus, if for any i $e_i \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi \neq 0$, $\lambda_i(u) = 1$ for every u belonging to P.

Hence the only such λ_i is the trivial character, whence we deduce that

$$\gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi = e_\chi \gamma e_\chi n_0 \varepsilon_{\phi_2} e_\psi.$$

Therefore we can assume that γ has the form $\sum_{n_i \in N} \mu_i n_i$ for some μ_i belonging to C.

For every h belonging to H,

$$h\, e_\chi n_0 \varepsilon_{\phi_1} e_\psi = \phi_1(h^{n_0})\, e_\chi n_0 \varepsilon_{\phi_1} e_\psi,$$

whence for every h belonging to H

$$h\, e_\chi \Big(\sum_{n_i \in N} \mu_i n_i\Big) e_\chi n_0 \varepsilon_{\phi_2} e_\psi = \phi_1(h^{n_0})\, e_\chi \Big(\sum_{n_i \in N} \mu_i n_i\Big) e_\chi n_0 \varepsilon_{\phi_2} e_\psi$$

Therefore, for every h belonging to H

$$e_{\chi}\Big(\sum_{n_i \in N} \mu_i \varphi_2(h^{n_i n_0})\, n_i\Big)\, e_{\chi}\, n_0\, \varepsilon_{\varphi_2} e_{\psi} = \varphi_1(h^{n_0})\, e_{\chi}\Big(\sum_{n_i \in N} \mu_i n_i\Big)\, e_{\chi}\, n_0\, \varepsilon_{\varphi_2} e_{\psi}$$

Hence there is exactly one non-zero μ_i, and for this i, $\varphi_2(h^{n_i n_0}) = \varphi_1(h^{n_0})$ for every h belonging to H.

Thus φ_1, φ_2 are conjugate in G.

The reverse implication is obvious.

<u>Corollary</u> The number of distinct, irreducible characters of G yielded by the construction described in lemma 6 is greater than or equal to the number of conjugacy classes of G which contain elements of H, i.e. the number of G-conjugacy classes of H.

<u>Lemma 9</u> If $G = GL(m,q)$ this lower bound is obtained, i.e. the number of irreducible characters of G obtainable by the method described is equal to the number of G-conjugacy classes of H.

<u>Proof</u> The number of ψ of general aspect is equal to $(q-1)^{m-1}$.

The normaliser of P is PH, and the order of H is $(q-1)^m$.

The order of the centraliser in H of P is $q-1$, whence it follows that all the ψ of general aspect are conjugate in G.

For the other groups under discussion it is not in general true that all ψ of general aspect are conjugate in G, or that the above lower bound is attained. This is shown by the following table and example

Group	Number of ψ of general aspect	$\lvert H \rvert$	$\lvert C_H(P) \rvert$	Number of 'distinct' ψ
$GL(m,q)$	$(q-1)^{m-1}$	$(q-1)^m$	$(q-1)$	1
$SL(m,q)$	$(q-1)^{m-1}$	$(q-1)^{m-1}$	$(m,q-1)$	$(m,q-1)$
$Sp(2m,q)$	$(q-1)^m$	$(q-1)^m$	$(2,q-1)$	$(2,q-1)$
$\Omega(2m,q)$	$(q-1)^m$	$(q-1)^m$	$(2,q-1)$	$(2,q-1)$
$\Omega(2m+1,q)$	$(q-1)^{m+1}$	$(q-1)^{m+1}$	$(2,q-1)(q-1)$	$(2,q-1)$

<u>Example</u> Let $G = SL(2,5)$

	R_1 $\begin{bmatrix}1 & \\ & 1\end{bmatrix}$	R_2 $\begin{bmatrix}-1 & \\ & -1\end{bmatrix}$	R_3 $\begin{bmatrix}t^6 & \\ & t^{-6}\end{bmatrix}$	R_4 $\begin{bmatrix}t^4 & \\ & t^{-4}\end{bmatrix}$	R_5 $\begin{bmatrix}t^8 & \\ & t^{-8}\end{bmatrix}$	R_6 $\begin{bmatrix}-1 & 1\\ & -1\end{bmatrix}$	R_7 $\begin{bmatrix}1 & 1\\ & 1\end{bmatrix}$	R_8 $\begin{bmatrix}-1 & t^6\\ & -1\end{bmatrix}$	R_9 $\begin{bmatrix}1 & t^6\\ & 1\end{bmatrix}$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	0	1	-1	-x	x	-y	y
χ_3	2	-2	0	1	-1	-y	y	-x	x
χ_4	3	3	-1	0	0	x	x	y	y
χ_5	3	3	-1	0	0	y	y	x	x
χ_6	4	-4	0	-1	1	1	-1	1	-1
χ_7	4	4	0	1	1	-1	-1	-1	-1
χ_8	5	5	1	-1	-1	0	0	0	0
χ_9	6	-6	0	0	0	-1	1	-1	1

where t is a generator of $GF(q^2)$, $x = \frac{1+\sqrt{5}}{2}$, $y = \frac{1-\sqrt{5}}{2}$

The conjugacy classes of G which contain elements of H are R_1, R_2, R_3.

The Sylow p-subgroup of G has order 5, whence it follows that P has four characters of general aspect. These latter are conjugate in pairs. If we denote the trivial character of P by ψ_1 and two of the non-conjugate, non-trivial characters by ψ_2, ψ_3 we find that

$$\psi_1^G = \chi_1 + \chi_4 + \chi_5 + \chi_8 + 2\chi_9$$
$$\psi_2^G = \chi_2 + \chi_4 + \chi_6 + \chi_7 + \chi_8 + \chi_9$$
$$\psi_3^G = \chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_8 + \chi_9 .$$

Hence the characters obtainable by the given method are χ_4, χ_5, χ_8, χ_9.

Thus not only is the lower bound of three not attained, but neither is the upper bound of six.

(upper bound = number of 'distinct' ψ . lower bound)

Although this example shows that, unless we do some rather heavy calculations, we are going to have to be satisfied with the above inequalities for the size of our family of characters, the table shows that the upper and lower bounds are not too disparate.

The next lemma shows that the well-known characters of Steinberg are always contained in our class.

Lemma 10 If φ is the trivial character of H, the character afforded by the module $CG\, e_{\chi} n_{o} \varepsilon_{\varphi} e_{\psi}$ is a Steinberg character.

Proof In (3) Curtis gave the following characterisation of the Steinberg character, namely, that it is the unique irreducible character of G which is a constituent of $(1_B)^G$ but not of $(1_{G_J})^G$ where G_J is a parabolic subgroup of G properly containing B.

In proving that the character afforded by the module $CG\, e_{\chi} n_{o} \varepsilon_{\varphi} e_{\psi}$ has this property, it is sufficient to consider those parabolic subgroups of the form

$G_G = B \cup Bn_r B$ where r is some fundamental root of the Lie algebra associated with G.

That the character under consideration is a constituent of $(1_B)^G$ is obvious.

Hence what we have to prove is that for every fundamental root r, $\mathrm{Hom.}_{CG}(CGf_r\, ,\ CG\, e_{\chi} n_{o} \varepsilon_{\varphi} e_{\psi}) = 0$, where f_r is the idempotent $\frac{1}{|G_J|}\left(\sum_{b\in B} b + \sum_{\substack{b\in B\\ t\in GF(q)}} b\, n_r x_r(t)\right)$

$\mathrm{Hom.}_{CG}(CGf_r\, ,\ CG\, e_{\chi} n_{o} \varepsilon_{\varphi} e_{\psi}) \cong f_r CG\, e_{\psi} n_{o} e_{\chi} n_{o} \varepsilon_{\varphi} e_{\psi}$

and hence it is sufficient to prove that $f_r CG\, e_{\psi} = 0$.

$$\varepsilon_{\varphi} e_{\chi} f_r = f_r = f_r \varepsilon_{\varphi} e_{\chi}$$

and $$\varepsilon_{\varphi} e_{\chi} CG\, e_{\psi} = C\, \varepsilon_{\varphi} e_{\chi} n_{o} e_{\psi}.$$

Hence $\quad f_r CG\, e_\psi = C\, \varepsilon_\varphi e_\chi f_r n_o e_\psi$.

$$\varepsilon_\varphi e_\chi f_r n_o e_\psi = \frac{1}{|G_J|} \varepsilon_\varphi e_\chi \Big(\sum_{b\in B} b + \sum_{\substack{b\in B \\ t\in GF(q)}} b\, n_r x_r(t) \Big) n_o e_\psi$$

$$= \frac{|B|}{|G_J|} \Big(\varepsilon_\varphi e_\chi n_o e_\psi + \sum_{t\in GF(q)} \varepsilon_\varphi e_\chi n_r x_r(t)\, n_o e_\psi \Big)$$

Thus we are home if we can show that

$$\sum_{t\in GF(q)} \varepsilon_\varphi e_\chi n_r x_r(t)\, n_o e_\psi = -\varepsilon_\varphi e_\chi n_o e_\psi \, .$$

We consider therefore the element $n_r x_r(t)\, n_o$.

$$n_r x_r(t)\, n_o = x_{-r}(\tau)\, n_r n_o \quad \text{for some } \tau \in GF(q)$$

There are now two cases to be considered

<u>Case 1</u> $\tau = 0$

$$n_r x_r(t)\, n_o = n_r n_o$$

This case makes no contribution to the above sum since

$e_\chi n_r n_o e_\psi = 0$.

<u>Case 2</u> $\tau \neq 0$

$$n_r x_r(t)\, n_o = b\, n_o x_{w_o(-r)}(\rho) \quad \text{for some } b \in B, \; \rho \in GF(q)^*$$

It is also true that different t give rise to different

ρ (This follows from the proof that $G = BNB$ see (2))

Hence $\sum_{t\in GF(q)} \varepsilon_\varphi e_\chi n_r x_r(t)\, n_o e_\psi = \sum_{t\in GF(q)^*} \varepsilon_\varphi e_\chi n_o x_{w_o(-r)}(t)\, e_\psi$.

Since r is a fundamental root, so also is $w_o(-r)$, whence

$$\sum_{t\in GF(q)^*} \varepsilon_\varphi e_\chi n_o x_{w_o(-r)}(t)\, e_\psi = \varepsilon_\varphi e_\chi n_o e_\psi \Big(\sum_{t\in GF(q)^*} \psi(x_{w_o(-r)}(t)) \Big)$$

$$= -\varepsilon_\varphi e_\chi n_o e_\psi \quad \text{as required}$$

Thus $\mathrm{Hom.}_{CG}(CGf_r \,,\, CG\, e_\chi n_o \varepsilon_\varphi e_\psi) = 0$, and lemma 10 follows from Curtis' theorem.

References

(1) E. Artin, C. Nesbitt and R. M. Thrall : Rings with minimum condition. Ann Arbor (1948).

(2) R. W. Carter : Lecture notes. University of Warwick 1968.

(3) C. W. Curtis : The Steinberg character of a finite group with a (B,N)-pair. J. of Alg. 4(1966) pp.433-441.

(4) I. M. Gelfand and M. I. Graev : Construction of irreducible representations of simple algebraic groups over a finite field. Dokl. Acad. Nauk. S.S.S.R. 147 (1962) pp. 529-532.

(5) D. E. Littlewood : The theory of group characters and matrix representations of groups. Oxford (1940).